

QUANTUM GROUPS WITH PARTIAL COMMUTATION RELATIONS

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ABSTRACT. We define new noncommutative spheres with partial commutation relations for the coordinates. We investigate the quantum groups acting maximally on them, which yields new quantum versions of the orthogonal group: They are partially commutative in a way such that they do *not* interpolate between the classical and the free quantum versions of the orthogonal group. Likewise we define non-interpolating, partially commutative quantum versions of the symmetric group recovering Bichon's quantum automorphism groups of graphs. They fit with the mixture of classical and free independence as recently defined by Speicher and Wysoczanski (rediscovering Λ -freeness of Mlotkowski), due to some weakened version of a de Finetti theorem.

1. INTRODUCTION

Motivated by the recent preprint on mixtures of classical and free independence by Wysoczanski and the first author [SW16] – where the notion of and results on Λ -freeness of Mlotkowski [Mł04] were rediscovered – we ask for the corresponding quantum symmetries. The mixture of independences goes as follows. Let $\varepsilon = (\varepsilon_{ij})_{i,j \in \{1, \dots, n\}}$ be a symmetric matrix with $\varepsilon_{ij} \in \{0, 1\}$ and $\varepsilon_{ii} = 0$. If variables x_1, \dots, x_n are ε -independent, then:

- x_i and x_j are free in the case $\varepsilon_{ij} = 0$
- and x_i and x_j are independent in the case $\varepsilon_{ij} = 1$ (in particular $x_i x_j = x_j x_i$ in this situation).

If all entries of ε are zero ($\varepsilon = \varepsilon_{\text{free}}$), we obtain Voiculescu's free independence; if all non-diagonal entries of ε are one ($\varepsilon = \varepsilon_{\text{comm}}$), we obtain classical independence.

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It is well-known that independences can be characterized via distributional symmetries with the help of de Finetti type theorems. For instance, classical independence is equivalent to invariance under actions of the symmetric group S_n , whereas free independence is characterized [KS09] by invariance under actions of Wang's [Wan98] quantum symmetric group S_n^+ . Our aim is to find the right quantum groups corresponding to the above ε -independence.

For doing so, we first define noncommutative ε -spheres $S_{\mathbb{R},\varepsilon}^{n-1}$ by:

$$C(S_{\mathbb{R},\varepsilon}^{n-1}) := C^*(x_1, \dots, x_n \mid x_i = x_i^*, \sum_i x_i^2 = 1, x_i x_j = x_j x_i, \text{ if } \varepsilon_{ij} = 1)$$

For $\varepsilon = \varepsilon_{\text{comm}}$ the above C^* -algebra is commutative and it is nothing but the algebra of continuous functions over the real sphere $S_{\mathbb{R}}^{n-1} \subset \mathbb{R}^n$. For $\varepsilon = \varepsilon_{\text{free}}$ we obtain Banica and Goswami's [BG10] free version $S_{\mathbb{R},+}^{n-1}$ of the sphere. In both cases we know the (quantum) group acting maximally on the sphere. In the first case, this is the orthogonal group $O_n \subset M_n(\mathbb{R})$ whereas in the second case [BG10], it is Wang's [Wan95] free orthogonal quantum group O_n^+ . We define an ε -version O_n^ε of these objects by

$$C(O_n^\varepsilon) = C^*(u_{ij}, i, j = 1, \dots, n \mid u_{ij} = u_{ij}^*, u \text{ is orthogonal, } R^\varepsilon \text{ hold}),$$

where R^ε are the relations:

$$u_{ik}u_{jl} = \begin{cases} u_{jl}u_{ik} & \text{if } \varepsilon_{ij} = 1 \text{ and } \varepsilon_{kl} = 1 \\ u_{jk}u_{il} & \text{if } \varepsilon_{ij} = 1 \text{ and } \varepsilon_{kl} = 0 \\ u_{il}u_{jk} & \text{if } \varepsilon_{ij} = 0 \text{ and } \varepsilon_{kl} = 1 \end{cases}$$

Moreover, we define an ε -version $S_n^\varepsilon \subset O_n^\varepsilon$ of the symmetric group by

$$C(S_n^\varepsilon) = C^*(u_{ij} \mid u_{ij} = u_{ij}^* = u_{ij}^2, \sum_k u_{ik} = \sum_k u_{kj} = 1 \forall i, j, \mathring{R}^\varepsilon \text{ hold}),$$

where \mathring{R}^ε are the relations:

$$u_{ik}u_{jl} = \begin{cases} u_{jl}u_{ik} & \text{if } \varepsilon_{ij} = 1 \text{ and } \varepsilon_{kl} = 1 \\ 0 & \text{if } \varepsilon_{ij} = 1 \text{ and } \varepsilon_{kl} = 0 \\ 0 & \text{if } \varepsilon_{ij} = 0 \text{ and } \varepsilon_{kl} = 1 \end{cases}$$

Note that the quotient of $C(S_n^+)$ by R^ε coincides with the one by \mathring{R}^ε (Lemma 6.2). Again, the choice $\varepsilon = \varepsilon_{\text{comm}}$ yields $O_n^\varepsilon = O_n$ and $S_n^\varepsilon = S_n$, whereas $\varepsilon = \varepsilon_{\text{free}}$ yields $O_n^\varepsilon = O_n^+$ and $S_n^\varepsilon = S_n^+$. The quantum groups S_n^ε coincide with Bichon's quantum automorphism groups of graphs [Bic03] and they are quantum subgroups of Banica's quantum automorphisms of graphs [Ban05]. The latter one are given by quotients of $C(S_n^+)$ by the relations $u\varepsilon = \varepsilon u$.

On the level of groups (or monoids) such mixed commutation relations have been studied extensively under names such as “right angled Artin groups”, “free partially commutative groups”, “trace groups”, “graph groups”, “Cartier-Foata monoids”, “trace monoids” etc, see for instance [Cha07, FC69] or the references in [SW16]. It is also linked to the following Coxeter groups (see Def 4.2):

$$\langle a_1, \dots, a_n \mid (a_i a_j)^{m_{ij}} = e \rangle, \quad m_{ij} = \begin{cases} 1 & \text{if } i = j \\ 2 & \text{if } \varepsilon_{ij} = 1 \\ \infty & \text{if } \varepsilon_{ij} = 0 \end{cases}$$

2. MAIN RESULTS

Our main result about the ε -sphere $S_{\mathbb{R}, \varepsilon}^{n-1}$ (or rather about its associated C^* -algebra $C(S_{\mathbb{R}, \varepsilon}^{n-1})$) is the following.

Theorem 2.1 (Thm. 4.5). *The ε -spheres are noncommutative as soon as $\varepsilon \neq \varepsilon_{\text{comm}}$ and we have $S_{\mathbb{R}, \varepsilon}^{n-1} \neq S_{\mathbb{R}, \varepsilon'}^{n-1}$ for $\varepsilon \neq \varepsilon'$. In particular,*

$$S_{\mathbb{R}}^{n-1} \subsetneq S_{\mathbb{R}, \varepsilon}^{n-1} \subsetneq S_{\mathbb{R}, +}^{n-1}$$

for $\varepsilon \neq \varepsilon_{\text{comm}}$ and $\varepsilon \neq \varepsilon_{\text{free}}$.

Here, $S_{\mathbb{R}, \varepsilon}^{n-1} \neq S_{\mathbb{R}, \varepsilon'}^{n-1}$ means that there is no isomorphism from $C(S_{\mathbb{R}, \varepsilon}^{n-1})$ to $C(S_{\mathbb{R}, \varepsilon'}^{n-1})$ mapping $x_i \mapsto x_i$. The results on O_n^ε may be summarized as follows.

Theorem 2.2 (Prop. 5.6, Thm. 5.7). *The quantum groups O_n^ε are no groups as soon as $\varepsilon \neq \varepsilon_{\text{comm}}$ and we have $O_n^\varepsilon \neq O_n^{\varepsilon'}$ for $\varepsilon \neq \varepsilon'$. The quantum group O_n^ε acts maximally on the ε -sphere.*

Note that while the ε -sphere $S_{\mathbb{R}, \varepsilon}^{n-1}$ interpolates the commutative sphere $S_{\mathbb{R}}^{n-1}$ and the free sphere $S_{\mathbb{R}, +}^{n-1}$, this is *not* the case for the ε -orthogonal quantum groups – if $\varepsilon \neq \varepsilon_{\text{comm}}$ and $\varepsilon \neq \varepsilon_{\text{free}}$, we have (see Prop. 5.3):

$$O_n \not\subseteq O_n^\varepsilon \subsetneq O_n^+ \quad \text{and} \quad S_n \not\subseteq O_n^\varepsilon \subsetneq O_n^+$$

Our original question about the symmetries of ε -independence is answered by the following weak version of a de Finetti theorem.

Theorem 2.3 (Thm. 9.6). *Let x_1, \dots, x_n be selfadjoint random variables in a noncommutative probability space (A, φ) such that $x_i x_j = x_j x_i$ if $\varepsilon_{ij} = 1$. If x_1, \dots, x_n are ε -independent and identically distributed, then their distribution is invariant under S_n^ε .*

We extend this de Finetti Theorem to the invariances by the quotients of H_n^+ , B_n^+ and O_n^+ by \mathring{R}^ε .

3. PRELIMINARIES ON ε -INDEPENDENCE

The notion of ε -independence as a mixture of classical and free independence has been introduced by the first author and Wysoczanski [SW16] very recently, rediscovering Mlotkowski's Λ -independence [Mł04]. We review its main features here. Throughout the whole article we denote by ε an $n \times n$ -matrix such that:

- $\varepsilon_{ij} \in \{0, 1\}$ for all $i, j = 1, \dots, n$
- ε is symmetric
- $\varepsilon_{ii} = 0$ for all $i = 1, \dots, n$

Definition 3.1. Let (A, φ) be a noncommutative probability space. We say that unital subalgebras $A_1, \dots, A_n \subset A$ are ε -independent, if we have:

- (i) The algebras A_i and A_j commute, if $\varepsilon_{ij} = 1$.
- (ii) Moreover, for any $k \in \mathbb{N}$ and any choice $a_1, \dots, a_k \in A$ with $a_j \in A_{i(j)}$ and the properties
 - $\varphi(a_j) = 0$ for all $j = 1, \dots, k$,
 - and for any $1 \leq p < r \leq k$ with $i(p) = i(r)$ there is a q with $p < q < r$ such that $\varepsilon_{i(p)i(q)} = 0$ and $i(p) \neq i(q)$,
 we have $\varphi(a_1 \cdots a_k) = 0$.

(Selfadjoint) variables $x_1, \dots, x_n \in A$ are ε -independent, if the algebras $\text{alg}(x_j, 1) \subset A$ are ε -independent.

Example 3.2. Here are a few examples of ε -independent variables.

- (a) Let $\varepsilon_{\text{comm}} \in M_n(\{0, 1\})$ be the matrix defined by $\varepsilon_{ij} = 1$ for all $i \neq j$ and $\varepsilon_{ii} = 0$ for all i . Variables x_1, \dots, x_n are ε -independent with respect to $\varepsilon_{\text{comm}}$ if and only if they all commute and are classically independent. Indeed, the constraint on the indices in Definition 3.1(ii) yields that all indices must be mutually different, in case $\varepsilon = \varepsilon_{\text{comm}}$. Now, by the usual centering trick on $a_j := x_{i(j)}^{m_j} - \varphi(x_{i(j)}^{m_j})$ with mutually different indices $i(j)$, we infer

$$\varphi(x_{i(1)}^{m_1} \cdots x_{i(k)}^{m_k}) = \prod_{j=1}^k \varphi(x_{i(j)}^{m_j})$$

and hence classical independence. See also [SW16, Prop. 3.2].

- (b) Let $\varepsilon_{\text{free}} \in M_n(\{0, 1\})$ be defined by $\varepsilon_{ij} = 0$ for all i, j . Variables x_1, \dots, x_n are ε -independent with respect to $\varepsilon_{\text{free}}$ if and only if they are freely independent. Note that if $\varepsilon = \varepsilon_{\text{free}}$, the constraint on the indices in Definition 3.1(ii) yields that neighbouring indices must be different. See also [SW16, Prop. 3.2].

(c) Let $\varepsilon \in M_{n+m}(\{0, 1\})$ be the matrix given by:

$$\varepsilon = \begin{pmatrix} \varepsilon_{\text{comm}} & 0 \\ 0 & 0 \end{pmatrix}, \quad \varepsilon_{ij} = \begin{cases} 1 & \text{if } i \leq n \text{ and } j \leq n \text{ and } i \neq j \\ 0 & \text{otherwise} \end{cases}$$

If variables $x_1, \dots, x_n, x_{n+1}, \dots, x_{n+m}$ are ε -independent with respect to this matrix, then

- (i) x_1, \dots, x_n are classically independent,
 - (ii) x_{n+1}, \dots, x_{n+m} are freely independent,
 - (iii) and $\{x_1, \dots, x_n\}$ and $\{x_{n+1}, \dots, x_{n+m}\}$ are free.
- (d) The iterated grouping of variables
- (i) x_1 and x_2 are independent,
 - (ii) x_3 and x_4 are independent,
 - (iii) and $\{x_1, x_2\}$ is free from $\{x_3, x_4\}$

is represented by the following matrix $\varepsilon \in M_4(\{0, 1\})$:

$$\varepsilon = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

(e) Swapping the terms “free” and “independent” in (d), we obtain:

$$\varepsilon = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

(f) A constellation that cannot be obtained from iterated grouping is the following (the motivating example in [SW16]):

- (i) x_i and x_{i+1} are free, for $i = 1, 2, 3, 4$, as well as x_5 and x_1 ,
- (ii) but all other pairs are independent.

The matrix $\varepsilon \in M_5(\{0, 1\})$ in this situation is:

$$\varepsilon = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix}$$

Like in the classical and the free case, we have a moment-cumulant formula for ε -independence. Let us first describe its combinatorics.

Definition 3.3. For $i = (i(1), \dots, i(k)) \in \{1, \dots, n\}^k$ we define $NC^\varepsilon[i]$ as the set of all partitions $\pi \in P(k)$ such that

- $\pi \leq \ker i$, i.e. if $1 \leq p < q \leq k$ are in the same block of π , then $i(p) = i(q)$,
- and π is (ε, i) -noncrossing, i.e. if there are indices $1 \leq p_1 < q_1 < p_2 < q_2 \leq k$ such that p_1 and p_2 are in a block V_p of π and q_1 and q_2 are in a block V_q of π with $V_p \neq V_q$, then $\varepsilon_{i(p_1)i(q_1)} = 1$.

The idea is, that $NC^\varepsilon[i]$ contains refinements of $\ker i$ which are allowed to have crossings only if the ε -entry of the crossing is 1.

Example 3.4. (a) For $\varepsilon = \varepsilon_{\text{comm}}$, all kinds of crossings between blocks on different indices are allowed, but not for different blocks on the same index. Hence we have:

$$NC^{\varepsilon_{\text{comm}}}[i] = \prod_{V \in \ker i} NC(V)$$

(b) For $\varepsilon = \varepsilon_{\text{free}}$, no crossings are allowed and hence:

$$NC^{\varepsilon_{\text{free}}}[i] = \{\pi \in NC(k) \mid \pi \leq \ker i\}$$

We now come to the moment-cumulant formula for ε -independence found by the first author and Wysoczanski. We only formulate it for the situation of ε -independent variables (rather than for algebras).

Proposition 3.5 ([SW16, Thm. 4.2]). *Let $x_1, \dots, x_n \in A$ be ε -independent and let $i = (i(1), \dots, i(k)) \in \{1, \dots, n\}^k$. Then:*

$$\varphi(x_{i(1)} \cdots x_{i(k)}) = \sum_{\pi \in NC^\varepsilon[i]} \kappa_\pi(x_{i(1)}, \dots, x_{i(k)})$$

Here, $\kappa_\pi(x_{i(1)}, \dots, x_{i(k)})$ is the product of the free cumulants for each block.

4. THE ε -SPHERE $S_{\mathbb{R}, \varepsilon}^{n-1}$

The sphere in \mathbb{R}^n (also called the *commutative sphere*) is given by:

$$S_{\mathbb{R}}^{n-1} = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid \sum_i x_i^2 = 1\}$$

The algebra of continuous functions over it may be written as a universal C^* -algebra:

$$C(S_{\mathbb{R}}^{n-1}) = C^*(x_1, \dots, x_n \mid x_i = x_i^*, \sum_i x_i^2 = 1, x_i x_j = x_j x_i \forall i, j)$$

A natural noncommutative analogue of the sphere is given by the (*maximally*) *noncommutative sphere* as introduced by Banica and Goswami

[BG10]:

$$C(S_{\mathbb{R},+}^{n-1}) := C^*(x_1, \dots, x_n \mid x_i = x_i^*, \sum_i x_i^2 = 1)$$

In the philosophy of noncommutative compact spaces, we may speak of the noncommutative sphere $S_{\mathbb{R},+}^{n-1}$ as a noncommutative compact space which is only defined via the algebra $C(S_{\mathbb{R},+}^{n-1})$ of (noncommutative) functions over it. Our next definition is an interpolation between the above spheres governed by the matrix ε .

Definition 4.1. The ε -sphere $S_{\mathbb{R},\varepsilon}^{n-1}$ is defined via the universal C^* -algebra:

$$C(S_{\mathbb{R},\varepsilon}^{n-1}) := C^*(x_1, \dots, x_n \mid x_i = x_i^*, \sum_i x_i^2 = 1, x_i x_j = x_j x_i, \text{ if } \varepsilon_{ij} = 1)$$

Note, that the commutative sphere $S_{\mathbb{R}}^{n-1}$ is an ε -sphere for the matrix $\varepsilon_{\text{comm}}$ of Example 3.2(a). Moreover, the noncommutative sphere $S_{\mathbb{R},+}^{n-1}$ is an ε -sphere with respect to $\varepsilon_{\text{free}}$ of Example 3.2(b). The ε -sphere is noncommutative, if $\varepsilon \neq \varepsilon_{\text{comm}}$, i.e. the C^* -algebra $C(S_{\mathbb{R},\varepsilon}^{n-1})$ is noncommutative in this case. We prove it by using representations of the ε -sphere which factor through the group C^* -algebras of certain Coxeter groups (see also [Ban15a, Ban16]).

Definition 4.2. Let \mathbf{F}_n be the free group with n generators a_1, \dots, a_n and denote by \mathbf{F}_n^ε the quotient of \mathbf{F}_n by the relations $a_i a_j = a_j a_i$ if $\varepsilon_{ij} = 1$. Denote by \mathbb{Z}_2^ε the quotient of \mathbf{F}_n^ε by the relations $a_i^2 = e$.

By $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ we denote the cyclic group of order two. We may view \mathbb{Z}_2^ε as the quotient of the n -fold free product \mathbb{Z}_2^{*n} by the relations $a_i a_j = a_j a_i$ if $\varepsilon_{ij} = 1$. It is a Coxeter group with the presentation:

$$\mathbb{Z}_2^\varepsilon = \langle a_1, \dots, a_n \mid (a_i a_j)^{m_{ij}} = e \rangle, \quad m_{ij} = \begin{cases} 1 & \text{if } i = j \\ 2 & \text{if } \varepsilon_{ij} = 1 \\ \infty & \text{if } \varepsilon_{ij} = 0 \end{cases}$$

Example 4.3. We have:

- (a) $\mathbb{Z}_2^\varepsilon = \mathbb{Z}_2 \times \dots \times \mathbb{Z}_2$ for $\varepsilon = \varepsilon_{\text{comm}}$ as in Example 3.2(a)
- (b) $\mathbb{Z}_2^\varepsilon = \mathbb{Z}_2 * \dots * \mathbb{Z}_2$ for $\varepsilon = \varepsilon_{\text{free}}$ as in Example 3.2(b)
- (c) $\mathbb{Z}_2^\varepsilon = \mathbb{Z}_2^{\times n} * \mathbb{Z}_2^{*m}$ for ε as in Example 3.2(c)
- (d) $\mathbb{Z}_2^\varepsilon = (\mathbb{Z}_2 \times \mathbb{Z}_2) * (\mathbb{Z}_2 \times \mathbb{Z}_2)$ for ε as in Example 3.2(d)
- (e) $\mathbb{Z}_2^\varepsilon = (\mathbb{Z}_2 * \mathbb{Z}_2) \times (\mathbb{Z}_2 * \mathbb{Z}_2)$ for ε as in Example 3.2(e)
- (f) For ε as in Example 3.2(f), we cannot write \mathbb{Z}_2^ε as an iteration of free and direct products.

The full group C^* -algebra associated to \mathbb{Z}_2^ε is the following universal C^* -algebra:

$$C^*(\mathbb{Z}_2^\varepsilon) = C^*(z_1, \dots, z_n \mid z_i = z_i^*, z_i^2 = 1, z_i z_j = z_j z_i \text{ if } \varepsilon_{ij} = 1)$$

Lemma 4.4. *Let H be a two-dimensional Hilbert space with orthonormal basis e_1 and e_2 .*

- (a) *For $n = 2$ and $\varepsilon = \varepsilon_{\text{free}}$, the Coxeter group $\mathbb{Z}_2^\varepsilon = \mathbb{Z}_2 * \mathbb{Z}_2$ may be represented on H by $\pi : C^*(\mathbb{Z}_2 * \mathbb{Z}_2) \rightarrow B(H)$ defined as:*

$$z_1 \mapsto a := \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad z_2 \mapsto b := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Note that a and b do not commute.

- (b) *Let $n \in \mathbb{N}$ and ε be arbitrary. Put $H_{ij} := H$ for $1 \leq i < j \leq n$. The representation $\sigma_\varepsilon : C(\mathbb{Z}_2^\varepsilon) \rightarrow B(\bigoplus_{i < j} H_{ij})$ given by*

$$\sigma_\varepsilon(z_k)|_{H_{ij}} = \begin{cases} a & \text{if } k = i \text{ and } \varepsilon_{ij} = 0 \\ b & \text{if } k = j \text{ and } \varepsilon_{ij} = 0 \\ \text{id}_H & \text{otherwise} \end{cases}$$

is such that $\sigma_\varepsilon(z_i)$ and $\sigma_\varepsilon(z_j)$ commute if and only if $\varepsilon_{ij} = 1$.

- (c) *We may represent the ε -sphere on the Coxeter group \mathbb{Z}_2^ε via:*

$$\varphi_\varepsilon : C(S_{\mathbb{R}, \varepsilon}^{n-1}) \rightarrow C^*(\mathbb{Z}_2^\varepsilon), \quad x_i \mapsto \frac{1}{\sqrt{n}} z_i$$

Proof. The proof is straightforward. \square

Theorem 4.5. *We have $S_{\mathbb{R}, \varepsilon}^{n-1} \neq S_{\mathbb{R}, \varepsilon'}^{n-1}$ for $\varepsilon \neq \varepsilon'$ in the sense that there is no $*$ -isomorphism $C(S_{\mathbb{R}, \varepsilon}^{n-1}) \rightarrow C(S_{\mathbb{R}, \varepsilon'}^{n-1})$ sending generators to generators. Moreover, $C(S_{\mathbb{R}, \varepsilon}^{n-1})$ is noncommutative as soon as $\varepsilon \neq \varepsilon_{\text{comm}}$.*

Proof. If $\varepsilon \neq \varepsilon'$, we may find indices i and j such that $\varepsilon_{ij} = 1$ and $\varepsilon'_{ij} = 0$ (possibly after swapping the names for ε and ε'). Assume that there is a $*$ -homomorphism $\psi : C(S_{\mathbb{R}, \varepsilon}^{n-1}) \rightarrow C(S_{\mathbb{R}, \varepsilon'}^{n-1})$ mapping generators to generators. Composing it with $\sigma_{\varepsilon'} \circ \varphi_{\varepsilon'}$ of the above lemma yields a contradiction, since x_i and x_j commute in $C(S_{\mathbb{R}, \varepsilon}^{n-1})$, but their images under $\sigma_{\varepsilon'} \circ \varphi_{\varepsilon'} \circ \psi$ do not. Noncommutativity of $C(S_{\mathbb{R}, \varepsilon}^{n-1})$ for $\varepsilon \neq \varepsilon_{\text{comm}}$ follows directly from applying $\sigma_\varepsilon \circ \varphi_\varepsilon$. \square

Corollary 4.6. *Let $\varepsilon \neq \varepsilon_{\text{comm}}$ and $\varepsilon \neq \varepsilon_{\text{free}}$. Seen as noncommutative compact spaces, we have:*

$$S_{\mathbb{R}}^{n-1} \subsetneq S_{\mathbb{R}, \varepsilon}^{n-1} \subsetneq S_{\mathbb{R}, +}^{n-1}$$

This means, we have surjective but non-injective $*$ -homomorphisms

$$C(S_{\mathbb{R}}^{n-1}) \leftarrow C(S_{\mathbb{R},\varepsilon}^{n-1}) \leftarrow C(S_{\mathbb{R},+}^{n-1})$$

sending generators to generators.

Remark 4.7. The study of noncommutative spheres has a long history and goes back to Podleś [Pod87]; see also the work of Connes with Dubois-Violette [CDV02] or with Landi [CL01], also collected in the survey [Lan05]; see also [Ban15a, Ban16] for recent expositions about noncommutative spheres and latest references.

The extensive work of Banica on noncommutative spheres is to be highlighted, see amongst others [BG10, Ban15a, Ban15b, Ban15c, Ban16]. However, his relations on the coordinates x_i are mostly chosen in a uniform way [Ban15b, Def. 2.2], [Ban15a, Def. 1.7] rather than as partial relations; so our spheres appear to be new.

5. THE ε -ORTHOGONAL QUANTUM GROUP O_n^ε

The algebra of functions over the orthogonal group $O_n \subset M_n(\mathbb{R})$ can be viewed as the following universal C^* -algebra:

$$C(O_n) = C^*(u_{ij}, i, j = 1, \dots, n \mid u_{ij} = u_{ij}^*, u \text{ is orthogonal}, R^{\text{comm}})$$

Here:

$$\begin{aligned} u \text{ is orthogonal} &\iff \sum_k u_{ik} u_{jk} = \sum_k u_{ki} u_{kj} = \delta_{ij} \\ R^{\text{comm}} &\iff u_{ij} u_{kl} = u_{kl} u_{ij} \quad \forall i, j, k, l \end{aligned}$$

Wang [Wan95] defined a noncommutative analogue of it, the (*free*) *orthogonal quantum group* O_n^+ given by:

$$C(O_n^+) = C^*(u_{ij}, i, j = 1, \dots, n \mid u_{ij} = u_{ij}^*, u \text{ is orthogonal})$$

For an introduction to compact matrix quantum groups, we refer to the original articles by Woronowicz [Wor87, Wor91] or the books [NT13, Tim08]. In the sequel, the tensor product of C^* -algebras is always with respect to the minimal tensor product.

Definition 5.1. We define the ε -orthogonal quantum group O_n^ε via the following universal C^* -algebra

$$C(O_n^\varepsilon) = C^*(u_{ij}, i, j = 1, \dots, n \mid u_{ij} = u_{ij}^*, u \text{ is orthogonal}, R^\varepsilon),$$

where the relations R^ε are defined by:

$$u_{ik} u_{jl} = \begin{cases} u_{jl} u_{ik} & \text{if } \varepsilon_{ij} = 1 \text{ and } \varepsilon_{kl} = 1 \\ u_{jk} u_{il} & \text{if } \varepsilon_{ij} = 1 \text{ and } \varepsilon_{kl} = 0 \\ u_{il} u_{jk} & \text{if } \varepsilon_{ij} = 0 \text{ and } \varepsilon_{kl} = 1 \end{cases}$$

We refer to Proposition 10.5 for further relations which are implied by the above ones.

Lemma 5.2. *The ε -orthogonal quantum group O_n^ε is a quantum group indeed, i.e. the C^* -algebra $C(O_n^\varepsilon)$ gives rise to a compact matrix quantum group in Woronowicz's sense.*

Proof. According to Woronowicz's axioms, all we have to prove is that the map $\Delta : C(O_n^\varepsilon) \rightarrow C(O_n^\varepsilon) \otimes C(O_n^\varepsilon)$ with $u_{ij} \mapsto u'_{ij} := \sum_k u_{ik} \otimes u_{kj}$ is a $*$ -homomorphism, i.e. that the elements $u'_{ij} \in C(O_n^\varepsilon) \otimes C(O_n^\varepsilon)$ satisfy the relations of the $u_{ij} \in C(O_n^\varepsilon)$. Self-adjointness and orthogonality of u' is easy to see, so it remains to show that the relations R^ε are fulfilled for the u'_{ij} . Consider first $\varepsilon_{ij} = 1$ and $\varepsilon_{kl} = 1$. Then we have:

$$\begin{aligned} u'_{ik} u'_{jl} &= \sum_{p,r: \varepsilon_{pr}=1} u_{ir} u_{jp} \otimes u_{rk} u_{pl} + \sum_{p,r: \varepsilon_{pr}=0} u_{ir} u_{jp} \otimes u_{rk} u_{pl} \\ &= \sum_{p,r: \varepsilon_{pr}=1} u_{jp} u_{ir} \otimes u_{pl} u_{rk} + \sum_{p,r: \varepsilon_{pr}=0} u_{jr} u_{ip} \otimes u_{rl} u_{pk} \\ &= \sum_{p,r} u_{jp} u_{ir} \otimes u_{pl} u_{rk} \\ &= u'_{jl} u'_{ik} \end{aligned}$$

Consider now $\varepsilon_{ij} = 1$ and $\varepsilon_{kl} = 0$. Then we have:

$$\begin{aligned} u'_{ik} u'_{jl} &= \sum_{p,r: \varepsilon_{pr}=1} u_{ir} u_{jp} \otimes u_{rk} u_{pl} + \sum_{p,r: \varepsilon_{pr}=0} u_{ir} u_{jp} \otimes u_{rk} u_{pl} \\ &= \sum_{p,r: \varepsilon_{pr}=1} u_{jp} u_{ir} \otimes u_{pk} u_{rl} + \sum_{p,r: \varepsilon_{pr}=0} u_{jr} u_{ip} \otimes u_{rk} u_{pl} \\ &= \sum_{p,r} u_{jp} u_{ir} \otimes u_{pk} u_{rl} \\ &= u'_{jk} u'_{il} \end{aligned}$$

The case $\varepsilon_{ij} = 0$ and $\varepsilon_{kl} = 1$ is similar. \square

Again, it is easy to see that O_n and O_n^+ fit into the framework of ε -orthogonal quantum groups, using the matrices $\varepsilon_{\text{comm}}$ and $\varepsilon_{\text{free}}$ respectively. However, let us point out that the commutativity relations R^{comm} do *not* imply R^ε for general ε , i.e. O_n^ε is *no interpolation* between O_n and O_n^+ . We say that a compact matrix quantum group G is a *quantum subgroup* of H (writing $G \subset H$), if there is a surjective $*$ -homomorphism from $C(H)$ to $C(G)$ mapping generators to generators.

Proposition 5.3. *If $\varepsilon \neq \varepsilon_{\text{comm}}$ and $\varepsilon \neq \varepsilon_{\text{free}}$, we have:*

$$O_n \not\subset O_n^\varepsilon \subsetneq O_n^+$$

More general, we have in that case:

$$S_n \not\subset O_n^\varepsilon \subsetneq O_n^+$$

Proof. Since the matrix u in O_n^ε is orthogonal and has self-adjoint entries, we have $O_n^\varepsilon \subset O_n^+$. The inclusion is strict, since $S_n \subset O_n \subset O_n^+$ but $S_n \not\subset O_n^\varepsilon$, which we will prove next. We may find $i \neq j$ such that $\varepsilon_{ij} = 1$ (since $\varepsilon \neq \varepsilon_{\text{free}}$), and $k \neq l$ such that $\varepsilon_{kl} = 0$ (since $\varepsilon \neq \varepsilon_{\text{comm}}$). Let $\sigma \in S_n$ be a permutation with the properties:

$$\sigma(k) = i, \quad \sigma(l) = j$$

The associated permutation matrix $a^\sigma \in M_n(\mathbb{C})$ is defined by $a_{pq}^\sigma = \delta_{p\sigma(q)}$. The evaluation map $\text{ev}_\sigma : C(S_n) \rightarrow \mathbb{C}$ is given by $\text{ev}_\sigma(u_{pq}) = \delta_{p\sigma(q)}$. Now, assume $S_n \subset O_n^\varepsilon$, i.e. there is a surjective $*$ -homomorphism $\varphi : C(O_n^\varepsilon) \rightarrow C(S_n)$ sending generators to generators. Composing it with ev_σ yields the following contradiction:

$$1 = \delta_{i\sigma(k)} \delta_{j\sigma(l)} = \text{ev}_\sigma \circ \varphi(u_{ik} u_{jl}) = \text{ev}_\sigma \circ \varphi(u_{jk} u_{il}) = \delta_{j\sigma(k)} \delta_{i\sigma(l)} = 0$$

□

Next, we will show that different matrices ε give rise to different ε -orthogonal quantum groups.

Lemma 5.4. *We have the following $*$ -homomorphism:*

$$\varphi_\varepsilon : C(O_n^\varepsilon) \rightarrow C^*(\mathbb{Z}_2^\varepsilon), \quad u_{ij} \mapsto \delta_{ij} z_i$$

Proof. The existence of π is due to the universal property. □

Remark 5.5. The preceding lemma is due to the fact that the diagonal subgroup of O_n^ε is the Coxeter group \mathbb{Z}_2^ε . The diagonal subgroup of a compact matrix quantum group (A, u) is constructed as follows. First, take the quotient of A by the relations $u_{ij} = 0$ for $i \neq j$. If u is a unitary, so are all u_{ii} in the quotient and we thus obtain the group C^* -algebra $C^*(G)$ of some group G . This group is called the diagonal subgroup of (A, u) .

Proposition 5.6. *We have $O_n^\varepsilon \neq O_n^{\varepsilon'}$ for $\varepsilon \neq \varepsilon'$ in the sense that there is no $*$ -isomorphism $C(O_n^\varepsilon) \rightarrow C(O_n^{\varepsilon'})$ sending generators to generators. Moreover, $C(O_n^\varepsilon)$ is noncommutative as soon as $\varepsilon \neq \varepsilon_{\text{comm}}$.*

Proof. The proof is similar to the one of Theorem 4.5 using the maps φ_ε of Lemma 5.4 rather than those of Lemma 4.4(c). □

Similarly to the well-known facts [BG10] that O_n acts maximally on the commutative sphere $S_{\mathbb{R}}^{n-1}$ and that O_n^+ acts maximally on the noncommutative sphere $S_{\mathbb{R},+}^{n-1}$, we observe that O_n^ε acts maximally on the ε -sphere $S_{\mathbb{R},\varepsilon}^{n-1}$.

Theorem 5.7. *The ε -orthogonal quantum group O_n^ε acts on the ε -sphere $S_{\mathbb{R},\varepsilon}^{n-1}$ by the natural left and right actions*

$$\alpha : C(S_{\mathbb{R},\varepsilon}^{n-1}) \rightarrow C(O_n^\varepsilon) \otimes C(S_{\mathbb{R},\varepsilon}^{n-1}), \quad x_i \mapsto \sum_k u_{ik} \otimes x_k$$

and:

$$\beta : C(S_{\mathbb{R},\varepsilon}^{n-1}) \rightarrow C(O_n^\varepsilon) \otimes C(S_{\mathbb{R},\varepsilon}^{n-1}), \quad x_i \mapsto \sum_k u_{ki} \otimes x_k$$

Moreover, O_n^ε is maximal with these actions in the sense that whenever G is a compact matrix quantum group acting on $S_{\mathbb{R},\varepsilon}^{n-1}$ in the above way, then $G \subset O_n^\varepsilon$.

Proof. Step 1: Existence of α and β .

We put $y_i := \sum_k u_{ik} \otimes x_k$ and compute for $\varepsilon_{ij} = 1$:

$$\begin{aligned} y_i y_j &= \sum_{k,l:\varepsilon_{kl}=1} u_{ik} u_{jl} \otimes x_k x_l + \sum_{k,l:\varepsilon_{kl}=0} u_{ik} u_{jl} \otimes x_k x_l \\ &= \sum_{k,l:\varepsilon_{kl}=1} u_{jl} u_{ik} \otimes x_l x_k + \sum_{k,l:\varepsilon_{kl}=0} u_{jk} u_{il} \otimes x_k x_l \\ &= \sum_{k,l} u_{jl} u_{ik} \otimes x_l x_k \\ &= y_j y_i \end{aligned}$$

Furthermore, $y_i^* = y_i$ and $\sum_i y_i^2 = 1$ by an easy computation using only the relations of O_n^+ . Thus, α exists by the universal property. Likewise we deduce the existence of β .

Step 2: Maximality; definition of auxiliary maps.

Now, let G be another compact matrix quantum group acting on $S_{\mathbb{R},\varepsilon}^{n-1}$ via:

$$\begin{aligned} \alpha', \beta' : C(S_{\mathbb{R},\varepsilon}^{n-1}) &\rightarrow C(G) \otimes C(S_{\mathbb{R},\varepsilon}^{n-1}) \\ \alpha'(x_i) &= \sum_k u_{ik} \otimes x_k, \quad \beta'(x_i) = \sum_k u_{ki} \otimes x_k \end{aligned}$$

For proving that there is a $*$ -homomorphism $C(O_n^\varepsilon) \rightarrow C(G)$ sending generators to generators, we will make use of the following $*$ -homomorphisms. They arise from tensor products of the identity map $\text{id} : C(G) \rightarrow C(G)$ with $*$ -homomorphisms from $C(S_{\mathbb{R},\varepsilon}^{n-1})$ to \mathbb{C} or to

$C(S_{\mathbb{R},+}^1)$ respectively; we use the universal property of $C(S_{\mathbb{R},\varepsilon}^{n-1})$ for the existence of the latter ones. We have:

$$\eta_k : C(G) \otimes C(S_{\mathbb{R},\varepsilon}^{n-1}) \rightarrow C(G), \quad z \otimes x_i \mapsto \begin{cases} z & \text{if } i = k \\ 0 & \text{otherwise} \end{cases}$$

Moreover, we have for $\varepsilon_{kl} = 1$:

$$\sigma_{kl} : C(G) \otimes C(S_{\mathbb{R},\varepsilon}^{n-1}) \rightarrow C(G), \quad z \otimes x_i \mapsto \begin{cases} \frac{1}{\sqrt{2}}z & \text{if } i = k \text{ or } i = l \\ 0 & \text{otherwise} \end{cases}$$

And for $\varepsilon_{kl} = 0$ with $k < l$:

$$\tau_{kl} : C(G) \otimes C(S_{\mathbb{R},\varepsilon}^{n-1}) \rightarrow C(G) \otimes C(S_{\mathbb{R},+}^1), \quad z \otimes x_i \mapsto \begin{cases} z \otimes x_1 & \text{if } i = k \\ z \otimes x_2 & \text{if } i = l \\ 0 & \text{otherwise} \end{cases}$$

Step 3: Maximality; $u_{ij} = u_{ij}^$ holds in $C(G)$.*

We observe that all generators of $C(G)$ are self-adjoint, by applying η_k to the following equation:

$$\sum_k u_{ik} \otimes x_k = \alpha'(x_i) = \alpha'(x_i)^* = \sum_k u_{ik}^* \otimes x_k$$

Step 4: Maximality; the relations R^ε hold in $C(G)$.

Let us compute:

$$\begin{aligned} \alpha'(x_i x_j) &= \sum_{k,l:\varepsilon_{kl}=1} u_{ik} u_{jl} \otimes x_k x_l + \sum_{k,l:\varepsilon_{kl}=0} u_{ik} u_{jl} \otimes x_k x_l \\ \alpha'(x_j x_i) &= \sum_{k,l:\varepsilon_{kl}=1} u_{jk} u_{il} \otimes x_k x_l + \sum_{k,l:\varepsilon_{kl}=0} u_{jk} u_{il} \otimes x_k x_l \\ \beta'(x_k x_l) &= \sum_{i,j:\varepsilon_{ij}=1} u_{ik} u_{jl} \otimes x_i x_j + \sum_{i,j:\varepsilon_{ij}=0} u_{ik} u_{jl} \otimes x_i x_j \\ \beta'(x_l x_k) &= \sum_{i,j:\varepsilon_{ij}=1} u_{il} u_{jk} \otimes x_i x_j + \sum_{i,j:\varepsilon_{ij}=0} u_{il} u_{jk} \otimes x_i x_j \end{aligned}$$

If $\varepsilon_{ij} = 1$, the terms $\alpha'(x_i x_j)$ and $\alpha'(x_j x_i)$ coincide. We apply τ_{kl} for $k < l$ and $\varepsilon_{kl} = 0$ to the equation $\alpha'(x_i x_j) = \alpha'(x_j x_i)$ and we obtain (where now $x_1, x_2 \in C(S_{\mathbb{R},+}^1)$):

$$\begin{aligned} &u_{ik} u_{jl} \otimes x_1 x_2 + u_{il} u_{jk} \otimes x_2 x_1 + u_{ik} u_{jk} \otimes x_1^2 + u_{il} u_{jl} \otimes x_2^2 \\ &= u_{jk} u_{il} \otimes x_1 x_2 + u_{jl} u_{ik} \otimes x_2 x_1 + u_{jk} u_{ik} \otimes x_1^2 + u_{jl} u_{il} \otimes x_2^2 \end{aligned}$$

By applying the maps η_1 and η_2 , we obtain $u_{ik} u_{jk} = u_{jk} u_{ik}$ and $u_{il} u_{jl} = u_{jl} u_{il}$. By Lemma 4.4(a) we know $x_1 x_2 \neq x_2 x_1$, so we finally obtain the

following relations (including the case $k = l$):

$$\varepsilon_{ij} = 1, \varepsilon_{kl} = 0 : \quad u_{ik}u_{jl} = u_{jk}u_{il}$$

A similar argument using β' yields:

$$\varepsilon_{ij} = 0, \varepsilon_{kl} = 1 : \quad u_{ik}u_{jl} = u_{il}u_{jk}$$

For $\varepsilon_{kl} = 1$, we have $x_kx_l = x_lx_k$, hence applying σ_{kl} to the equation $\alpha'(x_ix_j) = \alpha'(x_jx_i)$ yields the relations:

$$\varepsilon_{ij} = 1, \varepsilon_{kl} = 1 : \quad u_{ik}u_{jl} + u_{il}u_{jk} = u_{jk}u_{il} + u_{jl}u_{ik}$$

Applying it on $\beta'(x_kx_l) = \beta'(x_lx_k)$ yields the relations:

$$\varepsilon_{ij} = 1, \varepsilon_{kl} = 1 : \quad u_{ik}u_{jl} + u_{jk}u_{il} = u_{il}u_{jk} + u_{jl}u_{ik}$$

Combining these two relations, we obtain:

$$\varepsilon_{ij} = 1, \varepsilon_{kl} = 1 : \quad u_{ik}u_{jl} = u_{jl}u_{ik}$$

Step 5: Maximality; u is orthogonal in $C(G)$.

Using the relation $\sum_k x_k^2 = 1$ in $C(S_{\mathbb{R},\varepsilon}^{n-1})$, we infer:

$$\begin{aligned} 1 \otimes 1 &= \sum_k \alpha'(x_k^2) \\ &= \sum_{kij} u_{ki}u_{kj} \otimes x_ix_j \\ &= \sum_{ij:i \neq j} \left(\sum_k u_{ki}u_{kj} \right) \otimes x_ix_j + \sum_i \left(\sum_k u_{ki}^2 \right) \otimes x_i^2 \end{aligned}$$

Applying η_i to this equation, we obtain

$$\sum_k u_{ki}^2 = 1 \quad \forall i$$

and therefore:

$$\sum_{ij:i \neq j} \left(\sum_k u_{ki}u_{kj} \right) \otimes x_ix_j = 0$$

If now $\varepsilon_{ij} = 0$, applying τ_{ij} and using $x_1x_2 \neq x_2x_1$ in $C(S_{\mathbb{R},+}^1)$ yields:

$$\sum_k u_{ki}u_{kj} = 0$$

If $\varepsilon_{ij} = 1$, then $x_ix_j = x_jx_i$ and using σ_{ij} we deduce:

$$\sum_k u_{ki}u_{kj} + \sum_k u_{kj}u_{ki} = 0$$

But as $u_{ki}u_{kj} = u_{kj}u_{ki}$, we infer:

$$\sum_k u_{ki}u_{kj} = 0$$

Performing similar computations for β' , this proves orthogonality of u and we may conclude that there is a $*$ -homomorphism $C(G) \rightarrow C(O_n^\varepsilon)$ sending generators to generators; hence $G \subset O_n^\varepsilon$. \square

6. THE ε -SYMMETRIC QUANTUM GROUP S_n^ε

Having defined an ε -version of the orthogonal group O_n by quotienting out the relations R^ε from O_n^+ , it is natural to define ε -versions of quantum subgroups of O_n^+ in the same way. In order to do so for the symmetric (quantum) group, we observe that several natural relations are equivalent, as will be discussed in the sequel.

6.1. The relations \mathring{R}^ε . Recall the relations R^ε from Definition 5.1:

$$u_{ik}u_{jl} = \begin{cases} u_{jl}u_{ik} & \text{if } \varepsilon_{ij} = 1 \text{ and } \varepsilon_{kl} = 1 \\ u_{jk}u_{il} & \text{if } \varepsilon_{ij} = 1 \text{ and } \varepsilon_{kl} = 0 \\ u_{il}u_{jk} & \text{if } \varepsilon_{ij} = 0 \text{ and } \varepsilon_{kl} = 1 \end{cases}$$

We now define some simpler relations.

Definition 6.1. We define the relations \mathring{R}^ε by:

$$u_{ik}u_{jl} = \begin{cases} u_{jl}u_{ik} & \text{if } \varepsilon_{ij} = 1 \text{ and } \varepsilon_{kl} = 1 \\ 0 & \text{if } \varepsilon_{ij} = 1 \text{ and } \varepsilon_{kl} = 0 \\ 0 & \text{if } \varepsilon_{ij} = 0 \text{ and } \varepsilon_{kl} = 1 \end{cases}$$

In fact, we may also express them as:

$$\begin{aligned} (\mathring{R}^\varepsilon 1) \quad & u_{ik}u_{jl} = u_{jl}u_{ik} \text{ if } \varepsilon_{ij} = 1 \text{ and } \varepsilon_{kl} = 1 \\ (\mathring{R}^\varepsilon 2) \quad & \delta_{\varepsilon_{kl}=0} u_{ik}u_{jl} = \delta_{\varepsilon_{ij}=0} u_{ik}u_{jl} \end{aligned}$$

Lemma 6.2. *For any quantum subgroup $G_n \subset O_n^+$, we have:*

- (a) *The relations \mathring{R}^ε imply the relations R^ε .*
- (b) *If $u_{ik}u_{jk} = u_{ki}u_{kj} = 0$ for all $i \neq j$ and all k , then the relations R^ε imply the relations \mathring{R}^ε .*

Proof. (a) This follows since ε is symmetric.

(b) The relations $u_{ik}u_{jk} = 0$ imply that the elements u_{ik}^2 are projections (and thus, the u_{ik} are partial isometries). Indeed, we have $\sum_j u_{jk}^2 = 1$ by the orthogonality relations, thus:

$$u_{ik}^2 = u_{ik}^2 \sum_j u_{jk}^2 = u_{ik}^4 + \sum_{i \neq j} u_{ik}^2 u_{jk}^2 = u_{ik}^4$$

Recall that for $\varepsilon_{ij} = 1$ and $\varepsilon_{kl} = 0$, the relations R^ε imply:

$$u_{ik}u_{jl} = u_{jk}u_{il}$$

Multiplying this equation from the right with u_{jl}^2 , we infer

$$u_{ik}u_{jl} = 0,$$

since the projections u_{jl}^2 and u_{il}^2 are orthogonal to each other. The case $\varepsilon_{ij} = 0$ and $\varepsilon_{kl} = 1$ is similar. \square

Definition 6.3. For any quantum subgroup $G \subset O_n^+$ we define G^ε and \mathring{G}^ε by:

$$C(G^\varepsilon) := C(G)/\langle R^\varepsilon \rangle, \quad C(\mathring{G}^\varepsilon) := C(G)/\langle \mathring{R}^\varepsilon \rangle$$

For $\varepsilon = \varepsilon_{\text{free}}$, we have $G^\varepsilon = G$.

Lemma 6.4. *Let G be a compact matrix quantum group with fundamental unitary $u = (u_{ij})_{1 \leq i,j \leq n}$.*

- (a) *If the relations $(\mathring{R}^\varepsilon 2)$ hold for u_{ij} in $C(G)$, then they also hold for $u'_{ij} := \sum_k u_{ik} \otimes u_{kj} \in C(G) \otimes C(G)$.*
- (b) *If $G \subset O_n^+$ is a quantum subgroup of O_n^+ , then also G^ε and \mathring{G}^ε are compact matrix quantum groups and we have $\mathring{G}^\varepsilon \subset G^\varepsilon \subset G$.*

Proof. (a) We compute:

$$\begin{aligned} \delta_{\varepsilon_{kl}=0} u'_{ik} u'_{jl} &= \sum_{pq} u_{ip} u_{jq} \otimes \delta_{\varepsilon_{kl}=0} u_{pk} u_{ql} \\ &= \sum_{pq} u_{ip} u_{jq} \otimes \delta_{\varepsilon_{pq}=0} u_{pk} u_{ql} \\ &= \sum_{pq} \delta_{\varepsilon_{ij}=0} u_{ip} u_{jq} \otimes u_{pk} u_{ql} \\ &= \delta_{\varepsilon_{ij}=0} u'_{ik} u'_{jl} \end{aligned}$$

(b) In Lemma 5.2 we proved that the relations R^ε pass from u_{ij} to $u'_{ij} := \sum_k u_{ik} \otimes u_{kj}$. Thus, G^ε is a compact matrix quantum group. As for \mathring{G}^ε , we use (a) and Lemma 6.2. \square

By the same argument as in Proposition 5.3 we see that whenever $\varepsilon \neq \varepsilon_{\text{comm}}$ and $\varepsilon \neq \varepsilon_{\text{free}}$, we have:

$$S_n \not\subset G^\varepsilon \subset O_n^+$$

This is particularly interesting, since the concept of easy quantum groups, as developed by Banica and Speicher [BS09], provides a powerful approach for defining and studying quantum subgroups $G \subset O_n^+$, see also [RW15]. However, they come with the restriction $S_n \subset G \subset O_n^+$.

Thus, the ε -versions of easy quantum groups are a further step in the direction of understanding all quantum subgroups of O_n^+ .

6.2. Definition of S_n^ε . For S_n^+ the quotient by R^ε coincides with the one by \check{R}^ε , by Lemma 6.2. Hence, we define the ε -symmetric group S_n^ε as follows.

Definition 6.5. The ε -symmetric group S_n^ε is given by the quotient of S_n^+ by the relations \check{R}^ε , i.e.:

$$C(S_n^\varepsilon) = C^*(u_{ij} \mid u_{ij} = u_{ij}^* = u_{ij}^2, \sum_k u_{ik} = \sum_k u_{kj} = 1 \ \forall i, j \text{ and } \check{R}^\varepsilon)$$

Viewing $\varepsilon \in M_n(\{0, 1\})$ as the adjacency matrix of an undirected graph Γ_ε , we observe that our definition of S_n^ε coincides with the one of a quantum automorphism group of Γ_ε given by Bichon [Bic03, Bic04], see Proposition 6.8. In this sense, we may justify the definition S_n^ε intrinsically, i.e. as the quantum symmetry of some quantum space; exactly like we motivated our definition of O_n^ε as the quantum symmetry of the ε -sphere. There is another definition of a quantum automorphism group of a graph given by Banica [Ban05]. We denote it by $S_n^{\Gamma_\varepsilon}$ in order to keep the notations used in this article consistent.

Definition 6.6 ([Ban05]). Given an undirected graph Γ_ε with adjacency matrix $\varepsilon \in M_n(\{0, 1\})$, its *quantum automorphism group* $S_n^{\Gamma_\varepsilon}$ is defined via:

$$C(S_n^{\Gamma_\varepsilon}) := C^*(u_{ij} \mid u_{ij} = u_{ij}^* = u_{ij}^2, \sum_k u_{ik} = \sum_k u_{kj} = 1 \ \forall i, j \text{ and } u\varepsilon = \varepsilon u)$$

More explicitly, $u\varepsilon = \varepsilon u$ may be expressed as:

$$\sum_k \delta_{\varepsilon_{kl}=1} u_{ik} = \sum_j \delta_{\varepsilon_{ij}=1} u_{jl}$$

Lemma 6.7. Let $G \subset S_n^+$ be a quantum subgroup of S_n^+ . The following relations are equivalent:

- (i) The partial relations of R^ε : $u_{ik}u_{jl} = u_{jk}u_{il}$ and $u_{ki}u_{lj} = u_{kj}u_{li}$ if $\varepsilon_{ij} = 1$ and $\varepsilon_{kl} = 0$.
- (ii) The partial relations ($\check{R}^\varepsilon 2$) of \check{R}^ε : $\delta_{\varepsilon_{kl}=0} u_{ik}u_{jl} = \delta_{\varepsilon_{ij}=0} u_{ik}u_{jl}$.
- (iii) The relations $u\varepsilon = \varepsilon u$: $\sum_k \delta_{\varepsilon_{kl}=1} u_{ik} = \sum_j \delta_{\varepsilon_{ij}=1} u_{jl}$.

Proof. The equivalence of (i) and (ii) follows from the proof of Lemma 6.2. We now prove that (ii) implies (iii). From (ii) we infer:

$$\sum_{k,j} \delta_{\varepsilon_{kl}=1} \delta_{\varepsilon_{ij}=0} u_{ik}u_{jl} = 0 \quad \text{and} \quad \sum_{k,j} \delta_{\varepsilon_{kl}=0} \delta_{\varepsilon_{ij}=1} u_{ik}u_{jl} = 0$$

Thus, using $\sum_j u_{jl} = \sum_k u_{ik} = 1$, we have:

$$\sum_k \delta_{\varepsilon_{kl}=1} u_{ik} = \sum_{k,j} \delta_{\varepsilon_{kl}=1} \delta_{\varepsilon_{ij}=1} u_{ik} u_{jl} = \sum_j \delta_{\varepsilon_{ij}=1} u_{jl}$$

Conversely, assume that (iii) holds. We thus have:

$$\sum_{k'} \delta_{\varepsilon_{k'l}=0} u_{ik'} = 1 - \sum_{k'} \delta_{\varepsilon_{k'l}=1} u_{ik'} = 1 - \sum_{j'} \delta_{\varepsilon_{ij'}=1} u_{j'l} = \sum_{j'} \delta_{\varepsilon_{ij'}=0} u_{j'l}$$

Furthermore, $\varepsilon_{kl} = 0$ and $\varepsilon_{k'l} = 1$ implies $k \neq k'$ and hence $u_{ik} u_{ik'} = 0$. Likewise we see $\delta_{\varepsilon_{ij'}=0} \delta_{\varepsilon_{ij}=1} u_{j'l} u_{jl} = 0$. Therefore:

$$\begin{aligned} \delta_{\varepsilon_{kl}=0} u_{ik} u_{jl} &= \sum_{k'} \delta_{\varepsilon_{kl}=0} \delta_{\varepsilon_{k'l}=0} u_{ik} u_{ik'} u_{jl} + \sum_{k'} \delta_{\varepsilon_{kl}=0} \delta_{\varepsilon_{k'l}=1} u_{ik} u_{ik'} u_{jl} \\ &= \sum_{k'} \delta_{\varepsilon_{kl}=0} \delta_{\varepsilon_{k'l}=0} u_{ik} u_{ik'} u_{jl} \\ &= \sum_{j'} \delta_{\varepsilon_{kl}=0} \delta_{\varepsilon_{ij'}=0} u_{ik} u_{j'l} u_{jl} \\ &= \sum_{j'} \delta_{\varepsilon_{kl}=0} \delta_{\varepsilon_{ij'}=0} \delta_{\varepsilon_{ij}=0} u_{ik} u_{j'l} u_{jl} \end{aligned}$$

On the other hand:

$$\begin{aligned} \delta_{\varepsilon_{ij}=0} u_{ik} u_{jl} &= \sum_{j'} \delta_{\varepsilon_{ij}=0} \delta_{\varepsilon_{ij'}=0} u_{ik} u_{j'l} u_{jl} \\ &= \sum_{k'} \delta_{\varepsilon_{ij}=0} \delta_{\varepsilon_{k'l}=0} u_{ik} u_{ik'} u_{jl} \\ &= \sum_{k'} \delta_{\varepsilon_{ij}=0} \delta_{\varepsilon_{k'l}=0} \delta_{\varepsilon_{kl}=0} u_{ik} u_{ik'} u_{jl} \end{aligned}$$

This proves that (ii) holds. \square

The previous lemma and the next proposition comparing the two different definitions of quantum automorphism groups of [Bic03] and [Ban05] may also be found in [Ful06, Sect. 3.1].

Proposition 6.8. *The ε -symmetric quantum group S_n^ε coincides with the quantum automorphism group of Γ_ε as defined by Bichon, and it is a quantum subgroup of the quantum automorphism group $S_n^{\Gamma_\varepsilon}$ as defined by Banica.*

Proof. The relations (3.2) of Theorem 3.2 in [Bic03] are equivalent to $(\mathring{R}^\varepsilon 2)$ and his relations (3.3) are equivalent to $(\mathring{R}^\varepsilon 1)$. His relations (3.4)

follow from (3.2) and (3.3) using for $\varepsilon_{kl} = 1$:

$$\sum_{ij} \delta_{\varepsilon_{ij}=1} u_{ik} u_{jl} = \sum_{ij} u_{ik} u_{jl} = \left(\sum_i u_{ik} \right) \left(\sum_j u_{jl} \right) = 1$$

The assertion $S_n^\varepsilon \subset S_n^{\Gamma_\varepsilon}$ follows from the previous lemma. \square

6.3. Noncommutativity of $C(S_n^\varepsilon)$. Observe that the C^* -algebras $C(G^\varepsilon)$ may collapse to something very small and they might be commutative. This depends on the particular choice of the matrix ε as may be seen in the next two examples of S_n^ε . Note that when Banica, Bichon and others investigate graphs which have no quantum symmetry [BB07, BBC07a, BBC07c], this is exactly the same question: They ask whether or not $C(S_n^{\Gamma_\varepsilon})$ is commutative. Such investigations and concrete examples may also be found in [Ful06, Thm. 5.6.1 and Thm. 6.4.1].

Recall from Section 4 that we may view the full group C^* -algebra associated to $\mathbb{Z}_2 * \mathbb{Z}_2$ as a universal C^* -algebra. We now give a well-known alternative presentation.

Lemma 6.9. *The following universal C^* -algebras are isomorphic and noncommutative.*

- (a) $C^*(\mathbb{Z}_2 * \mathbb{Z}_2) = C^*(z_1, z_2, 1 \mid z_i = z_i^*, z_i^2 = 1, i = 1, 2)$
- (b) $C^*(p, q, 1 \mid p = p^* = p^2, q = q^* = q^2)$

Proof. The isomorphism between (a) and (b) is given by:

$$z_1 \mapsto 2p - 1, \quad z_2 \mapsto 2q - 1, \quad 1 \mapsto 1$$

Noncommutativity follows from Lemma 4.4(a). \square

The following example has also been treated by Bichon in [Bic03, Prop. 3.3].

Example 6.10. Let $\varepsilon \in M_4(\{0, 1\})$ be as in Example 3.2(d) given by:

$$\varepsilon = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

The C^* -algebra $C(S_4^\varepsilon)$ is noncommutative and hence S_4^ε is a quantum group which is not a group.

Proof. The following matrix (using Lemma 6.9) in $M_4(C^*(\mathbb{Z}_2 * \mathbb{Z}_2))$ gives rise to a representation of $C(S_4^\varepsilon)$ as may be verified directly:

$$\begin{pmatrix} p & 1-p & 0 & 0 \\ 1-p & p & 0 & 0 \\ 0 & 0 & q & 1-q \\ 0 & 0 & 1-q & q \end{pmatrix}$$

We thus have a surjection of $C(S_4^\varepsilon)$ onto $C^*(\mathbb{Z}_2 * \mathbb{Z}_2)$ which proves that $C(S_4^\varepsilon)$ is noncommutative. \square

Example 6.11. Let $\varepsilon \in M_4(\{0, 1\})$ be as in Example 3.2(e) given by:

$$\varepsilon = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

The C^* -algebra $C(S_4^\varepsilon)$ is commutative, and hence S_4^ε is a group with $S_4^\varepsilon \subset S_4$. The group is computed explicitly in Example 7.3(b).

Proof. Let $i, k, j, l \in \{1, 2, 3, 4\}$. We will show that u_{ik} and u_{jl} commute.

Case 1: $\varepsilon_{ij} = 1$ and $\varepsilon_{kl} = 1$. Then u_{ik} and u_{jl} commute due to the relations R^ε .

Case 2: $\varepsilon_{ij} = 1$ and $\varepsilon_{kl} = 0$. By the definition of the matrix ε , there are two indices $p \neq q$ such that $\varepsilon_{kp} = \varepsilon_{kq} = 1$ and by Case 1, we know that u_{ik} commutes with u_{jp} as well as with u_{jq} . Moreover, u_{ik} commutes with u_{jk} since their product is zero if $i \neq j$ as the projections in each row and each column are orthogonal to each other. Now, since $\varepsilon_{kk} = 0$, we have $p \neq k$ and $q \neq k$ showing that u_{ik} commutes with three entries of the j -th row. As the fourth entry may be expressed as a linear combination of 1 and the other three entries (recall that we have $\sum_m u_{jm} = 1$), we infer that u_{ik} commutes with all entries of the j -th row, in particular with u_{jl} .

Case 3: $\varepsilon_{ij} = 0$ and $\varepsilon_{kl} = 1$. The argument in Case 2 is symmetric.

Case 4: $\varepsilon_{ij} = 0$ and $\varepsilon_{kl} = 0$. Again we use the fact that there are two indices $p \neq q$ such that $\varepsilon_{kp} = \varepsilon_{kq} = 1$, which by Case 3 yields that u_{ik} commutes with u_{jp} as well as with u_{jq} . Now, u_{ik} commutes with u_{jk} for any i and j and we conclude as above that u_{ik} and u_{jl} commute.

We conclude that $C(S_4^\varepsilon)$ is commutative, and hence $S_4^\varepsilon \subset S_4$. \square

The above two examples show that it depends on the choice of ε whether $C(S_4^\varepsilon)$ is commutative or not. However, recall from Proposition 5.6 that $C(O_4^\varepsilon)$ is noncommutative in both examples.

7. THE COMMUTATIVE VERSION OF S_n^ε : THE GROUP T_n^ε

Given the fact that S_n^ε may be a group in certain cases, it might be interesting to determine it. We now associate a subgroup of S_n to any S_n^ε , regardless whether S_n^ε is a group or not. It is the commutative version of S_n^ε .

Definition 7.1. For a given $\varepsilon \in M_n(\{0, 1\})$ we define the following subgroup of S_n :

$$T_n^\varepsilon := \{\sigma \in S_n \mid \sigma \varepsilon \sigma^{-1} = \varepsilon\} \subset S_n$$

It is nothing but the automorphism group of the graph Γ_ε , since any permutation σ with $\sigma \varepsilon \sigma^{-1} = \varepsilon$ is a bijection between the vertices of the graph such that i and j form an edge of the graph if and only if $\sigma(i)$ and $\sigma(j)$ do. Thus, if a graph has no quantum symmetries in the sense of [BB07], then $S_n^{\Gamma_\varepsilon} = S_n^\varepsilon = T_n^\varepsilon$.

It turns out that while easy quantum groups are quantum subgroups of O_n^+ with the restriction of containing S_n , their ε -versions come with the restriction of containing T_n^ε .

Proposition 7.2. Viewed as a quantum group, T_n^ε arises from the quotient $C(S_n)/\langle R^\varepsilon \rangle$, or, in other words, as the quotient of $C(S_n^\varepsilon)$ by the commutativity of all generators u_{ij} . Hence, for all quantum groups $S_n \subset G \subset O_n^+$ we have:

$$T_n^\varepsilon \subset G^\varepsilon \subset O_n^+$$

If S_n^ε is a group, then $S_n^\varepsilon = T_n^\varepsilon$.

Proof. Since the C^* -algebra $C(S_n)/\langle R^\varepsilon \rangle$ carries a compact matrix quantum group structure and since it is commutative, it is isomorphic to $C(H)$, where H is some subgroup of S_n . It is given by all permutation matrices $a^\sigma \in S_n$ satisfying the relations R^ε , i.e. we have:

$$\text{For } \varepsilon_{ij} = 1 \text{ and } \varepsilon_{kl} = 0 : \quad \delta_{i\sigma(k)} \delta_{j\sigma(l)} = a_{ik}^\sigma a_{jl}^\sigma = a_{jk}^\sigma a_{il}^\sigma = \delta_{j\sigma(k)} \delta_{i\sigma(l)}$$

$$\text{For } \varepsilon_{ij} = 0 \text{ and } \varepsilon_{kl} = 1 : \quad \delta_{i\sigma(k)} \delta_{j\sigma(l)} = a_{ik}^\sigma a_{jl}^\sigma = a_{il}^\sigma a_{jk}^\sigma = \delta_{i\sigma(l)} \delta_{j\sigma(k)}$$

Let $\sigma \in H$ be a permutation and let $k \neq l$. Put $i := \sigma(k)$ and $j := \sigma(l)$ and assume $\varepsilon_{kl} = 1$. Then $\varepsilon_{ij} = 1$ since we would have a contradiction otherwise resulting from the relations R^ε . Likewise, $\varepsilon_{kl} = 0$ implies $\varepsilon_{ij} = 0$. We deduce that H consists exactly of all permutations $\sigma \in S_n$ such that:

$$\varepsilon_{\sigma(k)\sigma(l)} = 1 \quad \Longleftrightarrow \quad \varepsilon_{kl} = 1$$

Writing $\sigma \in S_n$ as the permutation matrix $a^\sigma \in M_n(\mathbb{C})$ with $a_{pq}^\sigma = \delta_{p\sigma(q)}$, we see that the (k, l) -th entry of $(a^\sigma)^{-1} \varepsilon a^\sigma$ is exactly $\varepsilon_{\sigma(k)\sigma(l)}$ which proves $H = T_n^\varepsilon$.

Finally, the natural quotient map from $C(S_n^+)/\langle R^\varepsilon \rangle$ to $C(S_n)/\langle R^\varepsilon \rangle$ is an isomorphism, if $C(S_n^+)/\langle R^\varepsilon \rangle$ is commutative. Thus $S_n^\varepsilon = T_n^\varepsilon$, if $C(S_n^\varepsilon)$ is commutative. \square

Example 7.3. We now study T_n^ε in certain examples.

- (a) We have $T_n^\varepsilon = S_n$ if and only if $\varepsilon = \varepsilon_{\text{comm}}$ or $\varepsilon = \varepsilon_{\text{free}}$. Indeed, by definition, T_n^ε consists of all possibilities to permute rows and columns with the same permutation, such that ε does not change. Now, if there are i, j, k, l such that $\varepsilon_{ij} = 1$ and $\varepsilon_{kl} = 0$ with $k \neq l$, then the permutation $\sigma \in S_n$ with $\sigma(i) = k$ and $\sigma(j) = l$ is not contained in T_n^ε . We conclude that T_n^ε is large in the extreme cases (maximally commutative and maximally noncommutative situations) and smaller otherwise.
- (b) In Examples 3.2(d) and 6.10 as well as in Examples 3.2(e) and 6.11, the group T_4^ε is given by the subgroup of S_4 generated by the transpositions $(1, 2)$ and $(3, 4)$ and the cyclic permutation $(1, 2, 3, 4)$. It has eight elements.

Check that permuting the first column to the k -th column implies a unique condition for where the second row is permuted to. We are then left with two possible choices for the permutation of the other indices, thus we have four times two possibilities in total.

- (c) The following matrix has trivial group T_6^ε :

$$\varepsilon = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix}$$

Indeed, observe that the number of units in a column gives the restriction that we may only permute the first and the second column, or the third and the fourth, and finally the fifth and the sixth. But permuting the first column to the second position, we would need to permute the third row to the fifth, which is not allowed by the above mentioned restriction. Continuing this argument, we infer that T_6^ε consists only of the neutral element.

8. INTERTWINERS FOR A DE FINETTI THEOREM

By Woronowicz's Tannaka-Krein result [Wor88], any compact matrix quantum group is completely determined by its intertwiners. See for instance [TW15] for an introduction to intertwiners and Tannaka-Krein

theory close to our setting. For the easy quantum groups of Banica and Speicher, the intertwiner space is spanned by maps which are indexed by partitions. Let us recall how we associate linear maps to partitions π in $P(k, l)$ having k upper points and l lower points, see [BS09] for details. For multi indices $i = (i(1), \dots, i(k))$ and $j = (j(1), \dots, j(l))$ with entries from $\{1, \dots, n\}$ we denote by $\ker(i, j)$ the partition in $P(k, l)$ obtained from connecting two points if and only if the entries of the multi index (i, j) coincide. This definition is an extension of the definition of $\ker i$.

Definition 8.1 ([BS09]). Let $n \in \mathbb{N}$. To a partition $\pi \in P(k, l)$ with $k, l \in \mathbb{N}_0$, we associate the linear map:

$$T_\pi : (\mathbb{C}^n)^{\otimes k} \rightarrow (\mathbb{C}^n)^{\otimes l}$$

$$e_{i(1)} \otimes \dots \otimes e_{i(k)} \mapsto \sum_{j(1), \dots, j(l)} \delta_{\pi \leq \ker(i, j)} e_{j(1)} \otimes \dots \otimes e_{j(l)}$$

Here, $(\mathbb{C}^n)^{\otimes 0} = \mathbb{C}$, by convention. For $\pi \in P(l) = P(0, l)$, we have:

$$T_\pi(1) = \sum_{j(1), \dots, j(l)} \delta_{\pi \leq \ker(j)} e_{j(1)} \otimes \dots \otimes e_{j(l)}$$

$$(T_\pi)^*(e_{j(1)} \otimes \dots \otimes e_{j(l)}) = \delta_{\pi \leq \ker(j)}$$

In our situation, we need a further linear map in order to describe our intertwiners.

Definition 8.2. For $n \in \mathbb{N}$ we define $R_\chi^1 : (\mathbb{C}^n)^{\otimes 2} \rightarrow (\mathbb{C}^n)^{\otimes 2}$ by:

$$R_\chi^1(e_i \otimes e_j) := \delta_{\varepsilon_{ij}=1} e_j \otimes e_i$$

Lemma 8.3. Let $G \subset O_n^+$ be a compact matrix quantum group. The map R_χ^1 is an intertwiner of G if and only if the relations \mathring{R}^ε hold in G .

Proof. We first compute:

$$u^{\otimes 2} R_\chi^1(e_l \otimes e_k) = u^{\otimes 2} (\delta_{\varepsilon_{kl}=1} e_k \otimes e_l)$$

$$= \sum_{i, j} \delta_{\varepsilon_{kl}=1} u_{ik} u_{jl} \otimes e_i \otimes e_j$$

And:

$$R_\chi^1 u^{\otimes 2}(e_l \otimes e_k) = \sum_{i, j} u_{jl} u_{ik} \otimes R_\chi^1(e_j \otimes e_i)$$

$$= \sum_{i, j} \delta_{\varepsilon_{ij}=1} u_{jl} u_{ik} \otimes e_i \otimes e_j$$

We infer that R^1_χ is an intertwiner (i.e. $R^1_\chi u^{\otimes 2} = u^{\otimes 2} R^1_\chi$) if and only if for all i, j, k, l :

$$\delta_{\varepsilon_{kl}=1} u_{ik} u_{jl} = \delta_{\varepsilon_{ij}=1} u_{jl} u_{ik}$$

These relations are equivalent to \mathring{R}^ε of Definition 6.1. \square

In the next section, we will prove de Finetti theorems for S_n^ε , H_n^ε , $\mathring{O}_n^\varepsilon$ and $\mathring{B}_n^\varepsilon$, which all contain the intertwiner R^1_χ as an essential ingredient of our proof. For $\varepsilon = \varepsilon_{\text{free}}$, these quantum groups are easy quantum groups in the sense of Banica and Speicher [BS09]. Their categories of partitions are as follows.

Definition 8.4. We define the following subsets of the set P of all partitions.

- (i) P_2 is the set of all pair partitions, i.e. each block of any partition $\pi \in P_2$ consists of exactly two points.
- (ii) $P_{1,2}$ is the set of all partitions $\pi \in P$, whose blocks consist either of one or of two points.
- (iii) P_{even} consists of all partitions $\pi \in P$ whose blocks consist of an even number of points.

Let $\mathcal{C}(k) \subset P(k)$ be a set of partitions. We define for any multi index i of length k :

$$NC_{\mathcal{C}}^\varepsilon[i] := \mathcal{C}(k) \cap NC^\varepsilon[i]$$

The category of partitions of S_n^+ is NC , the category of H_n^+ is $NC \cap P_{\text{even}}$, the category of B_n^+ is $NC \cap P_{1,2}$ and the category of O_n^+ is $NC \cap P_2$, see [BS09, Web13]. The sets of the above definition behave nicely with respect to taking subpartitions.

Definition 8.5. Let $\pi \in P(k)$ and $\sigma \in P(l)$ with $l \leq k$. Then σ is a *subpartition* of π , if

- (i) there are indices $1 \leq p \leq q \leq k$ with $q - p + 1 = l$ such that π restricted to the points $p, p+1, \dots, q$ coincides with σ ,
- (ii) and no point $p \leq s \leq q$ of π is in the same block as a point $1 \leq t < p$ or $q < t \leq k$.

Lemma 8.6. Let $\mathcal{C} \in \{P, P_2, P_{1,2}, P_{\text{even}}\}$, let $\pi \in \mathcal{C}$ and let σ be a subpartition of π . Then $\sigma \in \mathcal{C}$ and also $\pi' \in \mathcal{C}$, where π' is the partition obtained when removing σ from π .

Proof. The conditions of \mathcal{C} on the number of points in each block hold true for σ and π' . \square

We need the following technical lemma.

Lemma 8.7. *Let $\pi \in P(k)$ be a partition containing no non-trivial subpartitions and let π consist of at least two blocks. Then, there is an index $1 \leq l < k$ such that:*

- (i) *The point l belongs to a block V whereas $l + 1$ belongs to V' , and the blocks V and V' cross.*
- (ii) *We have $\min\{x \in V'\} < \min\{x \in V\}$.*

Proof. Firstly, observe that no block of π consists of a single point (otherwise it would form a subpartition) and that every block crosses with at least one other block (otherwise we would either find subpartitions between its legs, or the block would form a subpartition itself). Secondly, check that we may always find a block V_p containing indices $p_1 < p_2$ such that

- (1) there is an index $p_1 < s < p_2$ whose block crosses with V_p ,
- (2) and there are no two indices q_1, q_2 in a block $V_q \neq V_p$ with $q_1 < p_1 < q_2 < p_2$.

For instance, the block V containing the point 1 does the job, with $p_1 := 1$ and $p_2 := \max\{x \in V\}$.

Now, let V_p and p_1, p_2 be such that (1) and (2) are satisfied and $p_2 - p_1$ is minimal. Then $l := p_2 - 1$ is not in V_p by minimality of $p_2 - p_1$ and we have (ii) because of (2). Assume that (i) does not hold. Then, the block V containing l does not cross with V_p . Hence, there is at least a second point $s \in V$ with $p_1 < s < l < p_2$ such that there is an index $s < t < l$ whose block crosses with V . Let $V_{p'} \neq V_p$ be the unique block containing indices s and u with $p_1 < s < u < p_2$ such that there is an index $s < t < u$ whose block crosses with $V_{p'}$ and such that $\min\{x \in V_{p'}\}$ is minimal. Then, $p'_1 := \min\{x \in V_{p'}\}$ and $p'_2 := \max\{x \in V_{p'} \mid x < p_2\}$ satisfy (1) and (2), but $p'_2 - p'_1 < p_2 - p_1$ contradicts the minimality assumption on V_p . \square

We are ready to prove the crucial ingredient for our de Finetti theorem. Note that for $\varepsilon = \varepsilon_{\text{free}}$, the proof is trivial.

Proposition 8.8. *Let $k \in \mathbb{N}$ and $\pi \in P(k)$. Let $G \subset O_n^+$ be a quantum subgroup of O_n^+ and let $\mathcal{C} \in \{P, P_2, P_{1,2}, P_{\text{even}}\}$. Let R_χ^1 be an intertwiner of G as well as the maps T_σ for $\sigma \in \mathcal{C} \cap NC$.*

- (a) *Let $\pi \in \mathcal{C}$. Then the following map is an intertwiner of G :*

$$M_\pi : (\mathbb{C}^n)^{\otimes k} \rightarrow \mathbb{C}, \quad e_{i(1)} \otimes \dots \otimes e_{i(k)} \mapsto \delta_{\pi \in NC_\varepsilon^1[i]}$$

- (b) *The following relations hold in G , for all $k \in \mathbb{N}$ and all $j(1), \dots, j(k) \in \{1, \dots, n\}$.*

$$\sum_{i(1), \dots, i(k)} \delta_{\pi \in NC_{\mathcal{C}}^{\varepsilon}[i]} u_{i(1)j(1)} \cdots u_{i(k)j(k)} = \delta_{\pi \in NC_{\mathcal{C}}^{\varepsilon}[j]}$$

Proof. Let $\pi \in \mathcal{C}(k)$ be a partition. We now construct the linear map M_{π} recursively from composing intertwiners of G . For doing so, we use the following algorithm to construct partitions $\pi_m \in P(k_m)$ and maps $M_m : (\mathbb{C}^n)^{\otimes k_m} \rightarrow \mathbb{C}$, for $m \in \mathbb{N}_0$.

Step 1: The algorithm for defining M_{π} .

Let $\pi_0 := \pi$ and $k_0 := k$ and begin the algorithm with $m = 0$.

- (Case 1) If there is a noncrossing subpartition σ of π_m on the indices $p, p+1, \dots, q$ with $1 \leq p \leq q \leq k_m$, we define

$$M_m := \text{id}^{\otimes p-1} \otimes T_{\sigma}^* \otimes \text{id}^{\otimes k_m-q}$$

and let π_{m+1} be the partition resulting from π_m when removing σ . Put $k_{m+1} := k_m - (q - p + 1)$. If $\sigma = \pi_m$, terminate the algorithm after this step.

- (Case 2) If Case 1 does not apply, let $1 \leq l < k_m$ be the smallest index such that:

- The point l belongs to a block V whereas $l+1$ belongs to V' , and the blocks V and V' cross.
- We have $\min\{x \in V'\} < \min\{x \in V\}$.

We put:

$$M_m := \text{id}^{\otimes (l-1)} \otimes R_{\chi}^1 \otimes \text{id}^{\otimes (k_m-l-1)}$$

We define π_{m+1} as the partition obtained from π_m when swapping the legs on l and $l+1$. We put $k_{m+1} := k_m$.

An example of the algorithm can be found in Figure 1. Note that if π_m does not contain a noncrossing subpartition, either π_m or one of its subpartitions satisfies the assumptions of Lemma 8.7 which ensures the existence of an index l as in Case 2.

Moreover, the algorithm terminates since coming from Case 2, we will either be in Case 1 in the next step (reducing the length of the partition, or terminating) or we will be again in Case 2 successively pulling two crossing blocks side by side, which eventually brings us back to Case 1. Thus, we obtain a finite number of maps M_0, \dots, M_t and we put:

$$M_{\pi} := M_t \dots M_0$$

Step 2: M_{π} is an intertwiner of G .

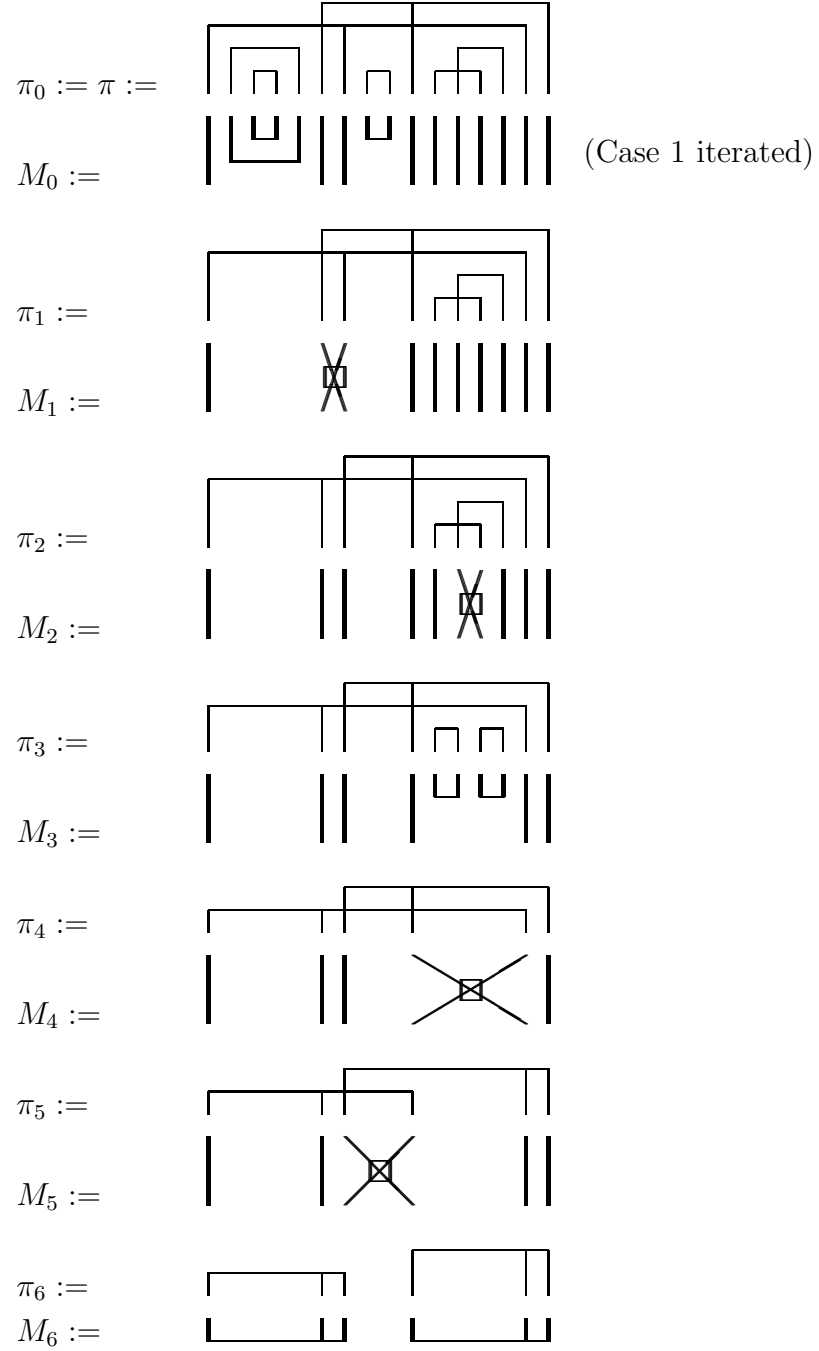


FIGURE 1. An example of the algorithm in Thm. 8.8.

It is easy to see, by Lemma 8.6, that $\pi_m \in \mathcal{C}$ if and only if $\pi_{m+1} \in \mathcal{C}$. Hence, since $\pi_0 = \pi$ is in \mathcal{C} , so are all π_m and also all of their subpartitions. Therefore, by assumption on the intertwiner space of G , all maps M_m are intertwiners of G , and so is M_π .

Step 3: Proof of $M_\pi e_i = \delta_{\pi \in NC_{\mathcal{C}}^\varepsilon[i]}$.

We are left with proving the formula:

$$M_\pi(e_{i(1)} \otimes \dots \otimes e_{i(k)}) = \delta_{\pi \in NC_{\mathcal{C}}^\varepsilon[i]}$$

Let us abbreviate $e_i := e_{i(1)} \otimes \dots \otimes e_{i(k)}$ for multi indices i . We are going to prove the following statement:

- (*) For all $0 \leq m \leq t-1$ and for all multi indices i , we have $\pi_m \in NC_{\mathcal{C}}^\varepsilon[i]$ if and only if $M_m e_i \neq 0$ and $\pi_{m+1} \in NC_{\mathcal{C}}^\varepsilon[j]$ for $e_j = M_m e_i$.

Having proven (*), we infer inductively that $\pi = \pi_0 \in NC_{\mathcal{C}}^\varepsilon[i]$ if and only if $\pi_t \in NC_{\mathcal{C}}^\varepsilon[j]$ for $e_j = M_{t-1} \dots M_0 e_i \neq 0$. Since the above algorithm terminates at step t , the partition π_t is noncrossing and hence $\pi_t \in NC_{\mathcal{C}}^\varepsilon[j]$ if and only if $M_t e_j = T_{\pi_t}^* e_j = 1$ and $\pi_t \in \mathcal{C}$. We conclude $\pi \in NC_{\mathcal{C}}^\varepsilon[i]$ if and only if $M_\pi e_i = 1$. Since $M_\pi e_i$ only takes the values zero or one, this proves $M_\pi e_i = \delta_{\pi \in NC_{\mathcal{C}}^\varepsilon[i]}$.

We are now going to prove (*). Let $0 \leq m \leq t-1$ and let i be a multi index.

Step 4: Proof of () with π_{m+1} resulting from Case 1 of the algorithm.*

Assume $\pi_m \in NC_{\mathcal{C}}^\varepsilon[i]$. Then $\pi_m \leq \ker i$ implies that the noncrossing subpartition σ of π_m is less or equal to the relevant section of the multi index i , and hence $M_m e_i \neq 0$. Since π_{m+1} arises as a restriction of π_m while j arises as a restriction of i , we also have $\pi_{m+1} \in NC_{\mathcal{C}}^\varepsilon[j]$.

Conversely, let $\pi_{m+1} \in NC_{\mathcal{C}}^\varepsilon[j]$ and $M_m e_i \neq 0$. We have $\pi_m \leq \ker i$ by the following. If p and q are in the same block of π_m , then either both of them are in j or none of them is, since σ is a subpartition. In the first case, $\pi_{m+1} \leq \ker j$ implies $i(p) = i(q)$ whereas in the second, this is ensured by $M_m e_i \neq 0$. Finally, π_m is (ε, i) -noncrossing since any crossing in π_m yields a crossing in π_{m+1} whose ε -entry is 1, because π_{m+1} is (ε, j) -noncrossing.

Step 5: Proof of () with π_{m+1} resulting from Case 2 of the algorithm.*

In Case 2 of the algorithm, π_{m+1} is obtained from π_m by swapping the legs on l and $l+1$. We have $M_m e_i \neq 0$ if and only if $\varepsilon_{i(l)i(l+1)} = 1$. Moreover, assuming $\pi_m \in NC_{\mathcal{C}}^\varepsilon[i]$, we obtain $\varepsilon_{i(l)i(l+1)} = 1$, since the blocks on l and $l+1$ cross. We may thus assume $\varepsilon_{i(l)i(l+1)} = 1$ throughout Step 5. We know that j is of the form:

$$j = (i(1), \dots, i(l-1), i(l+1), i(l), i(l+2), \dots, i(k_m))$$

Assume $\pi_m \leq \ker i$. We want to prove $\pi_{m+1} \leq \ker j$. Let p and q be in the same block of π_{m+1} . If $\{p, q\} \cap \{l, l+1\} = \emptyset$, then π_{m+1} and π_m coincide on p and q , i.e. p and q are also in the same block of π_m , implying $j(p) = i(p) = i(q) = j(q)$. On the other hand, if $\{p, q\} \cap \{l, l+1\} \neq \emptyset$, assume $p = l$. Then $q \neq l+1$ since l and $l+1$ are in different blocks. Now, l and q being in the same block of π_{m+1} implies that $l+1$ and q are in the same block of π_m and hence $j(p) = i(l+1) = i(q) = j(q)$. The other cases of $\{p, q\} \cap \{l, l+1\} \neq \emptyset$ are similar. Since the argument is symmetric, we just proved $\pi_m \leq \ker i$ if and only if $\pi_{m+1} \leq \ker j$.

Assume that π_m is (ε, i) -noncrossing. Then π_{m+1} is (ε, j) -noncrossing due to the following discussion. Let $p_1 < q_1 < p_2 < q_2$ be points of π_{m+1} such that $p_1, p_2 \in V_p$ and $q_1, q_2 \in V_q \neq V_p$. If $\{p_1, q_1, p_2, q_2\} \cap \{l, l+1\} = \emptyset$, we have $\varepsilon_{j(p_1)j(q_1)} = 1$ since π_{m+1} coincides with π_m on the relevant points. If $\{p_1, q_1, p_2, q_2\} \cap \{l, l+1\} = \{l\}$, then we are in the situation that some block V of π_{m+1} crosses with the block on l . This is equivalent to this block V of π_m crossing with the block on $l+1$ in π_m . We infer $\varepsilon_{j(p_1)j(q_1)} = 1$. We argue analogously if $\{p_1, q_1, p_2, q_2\} \cap \{l, l+1\} = \{l+1\}$. Finally, $\{p_1, q_1, p_2, q_2\} \cap \{l, l+1\} = \{l, l+1\}$ implies $\varepsilon_{i(p_1)i(q_1)} = \varepsilon_{i(l)i(l+1)} = 1$. Again, the argument is symmetric in π_m and π_{m+1} .

We conclude that $\pi_m \in NC_{\mathcal{C}}^{\varepsilon}[i]$ if and only if $\pi_{m+1} \in NC_{\mathcal{C}}^{\varepsilon}[j]$, provided that $\varepsilon_{i(l)i(l+1)} = 1$.

(b) Finally, $M_{\pi} u^{\otimes k} = M_{\pi}$ yields the desired relations on the u_{ij} 's. \square

9. SYMMETRIES OF ε -INDEPENDENCE

There are classical and noncommutative versions of de Finetti theorems characterizing independences by distributional symmetries. We recall when a distribution is invariant under a quantum group action. For details see [KS09, BCS12].

Definition 9.1. Let $x_1, \dots, x_n \in A$ be self-adjoint random variables in a noncommutative probability space (A, φ) .

- (a) Let $G \subset O_n^+$ be a compact matrix quantum group. We say that *the distribution of x_1, \dots, x_n is invariant under G* , if for all $k \in \mathbb{N}$ and all $j(1), \dots, j(k) \in \{1, \dots, n\}$:

$$\varphi(x_{j(1)} \cdots x_{j(k)})1 = \sum_{i(1), \dots, i(k)} \varphi(x_{i(1)} \cdots x_{i(k)}) u_{i(1)j(1)} \cdots u_{i(k)j(k)}$$

- (b) We say that the variables x_1, \dots, x_n are *identically distributed*, if we have for all $k \in \mathbb{N}$ and all $1 \leq i, j \leq n$:

$$\varphi(x_i^k) = \varphi(x_j^k)$$

The above definition (a) is a natural extension of the notion of distributional invariance for groups. Indeed, for instance if $G = S_n$, evaluating the above equation at $\sigma \in S_n$ yields:

$$\begin{aligned} \varphi(x_{j(1)} \cdots x_{j(k)})1 &= \sum_{i(1), \dots, i(k)} \varphi(x_{i(1)} \cdots x_{i(k)}) \delta_{i(1)\sigma(j(1))} \cdots \delta_{i(k)\sigma(j(k))} \\ &= \varphi(x_{\sigma(j(1))} \cdots x_{\sigma(j(k))})1 \end{aligned}$$

This is just the well-known exchangeability. We now recall some existing de Finetti theorems.

Proposition 9.2 ([KS09]). *Let $(x_n)_{n \in \mathbb{N}}$ be a sequence of selfadjoint random variables in a noncommutative W^* -probability space (M, φ) such that M is generated by $x_n, n \in \mathbb{N}$. The following holds true.*

- (a) *Suppose that the elements x_n commute. The sequence $(x_n)_{n \in \mathbb{N}}$ is conditionally independent and identically distributed if and only if its distribution is invariant under $(S_n)_{n \in \mathbb{N}}$.*
- (b) *The sequence $(x_n)_{n \in \mathbb{N}}$ is conditionally free and identically distributed if and only if its distribution is invariant under $(S_n^+)_{n \in \mathbb{N}}$.*

The above theorem has been extended to other quantum groups by Banica, Curran and the first author [BCS12]:

- Invariance under H_n^+ adds the condition that the distribution of the variables is even.
- Invariance under B_n^+ adds the condition that the distribution are semicircles with common mean and common variance.
- Invariance under O_n^+ adds the condition that the distribution are semicircles with mean zero and common variance.

In our framework, we do not have such a de Finetti theorem for the moment since we are lacking an operator-valued version of ε -independence (needed to formulate what “conditionally ε -independent” is supposed to mean). However, the equivalences of the above de Finetti theorems rely on a finite and purely algebraic version of one of the directions which we formulate here in the scalar-valued form.

Proposition 9.3 ([BCS12]). *Let x_1, \dots, x_n be selfadjoint random variables in a noncommutative probability space (A, φ) .*

- (a) *Suppose that the elements x_1, \dots, x_n commute. If x_1, \dots, x_n are independent and identically distributed, then their distribution is invariant under S_n .*
- (b) *If x_1, \dots, x_n are free and identically distributed, then their distribution is invariant under S_n^+ .*

Again, this statement has versions for H_n^+ , B_n^+ and O_n^+ , [BCS12]. We now prove an ε -version containing the above proposition as a special case. For doing so, we need to define further quantum groups based on Definition 6.3 and Lemma 6.4. Recall from Lemma 6.2 that the relations R^ε and \mathring{R}^ε are equivalent in any quotient of $C(H_n^+)$.

Definition 9.4. The ε -hyperoctahedral group H_n^ε is given by the quotient of H_n^+ by the relations \mathring{R}^ε , i.e.:

$$C(H_n^\varepsilon) := C^*(u_{ij} \mid u_{ik}u_{jk} = u_{ki}u_{kj} = 0 \ \forall i \neq j \text{ and } \mathring{R}^\varepsilon)$$

For $\varepsilon = \varepsilon_{\text{free}}$, the quantum group H_n^ε is nothing but the hyperoctahedral quantum group H_n^+ as defined by Banica, Bichon and Collins [BBC07b]. As for the analogs of O_n^+ and B_n^+ in our de Finetti theorem, there is a little subtlety: The relations R^ε and \mathring{R}^ε are *not* equivalent for general subgroups of O_n^+ . Moreover, our Proposition 8.8 requires the existence of R^1_\times as an intertwiner. Therefore (see Lemma 8.3), our de Finetti theorem is designed for quantum groups satisfying the relations \mathring{R}^ε .

Definition 9.5. We define the quantum group $\mathring{O}_n^\varepsilon$ via the following universal C^* -algebra:

$$C(\mathring{O}_n^\varepsilon) = C^*(u_{ij}, i, j = 1, \dots, n \mid u_{ij} = u_{ij}^*, u \text{ is orthogonal, } \mathring{R}^\varepsilon)$$

We define the quantum group $\mathring{B}_n^\varepsilon$ via the quotient of $C(\mathring{O}_n^\varepsilon)$ by the relations $\sum_k u_{ik} = \sum_k u_{kj} = 1$.

Note that for $\varepsilon = \varepsilon_{\text{free}}$, we have $\mathring{O}_n^\varepsilon = O_n^\varepsilon = O_n^+$, but for $\varepsilon = \varepsilon_{\text{comm}}$, we have $\mathring{O}_n^\varepsilon = H_n \subsetneq O_n = O_n^\varepsilon$, since in that case $u_{ik}u_{jk} = u_{ki}u_{kj} = 0$ in $\mathring{O}_n^\varepsilon$ for all $i \neq j$.

Theorem 9.6. Let $\varepsilon \in M_n(\{0, 1\})$ and let x_1, \dots, x_n be selfadjoint random variables in a noncommutative probability space (A, φ) such that $x_i x_j = x_j x_i$ if $\varepsilon_{ij} = 1$. Let x_1, \dots, x_n be ε -independent and identically distributed.

- (a) Then their distribution is invariant under S_n^ε .
- (b) If their distribution is even (all odd free cumulants vanish), then it is invariant under H_n^ε .
- (c) If only free cumulants of blocks of size one or two are non-zero (i.e. if the distribution is a shifted semicircular), then the distribution is invariant under $\mathring{B}_n^\varepsilon$.
- (d) If only free cumulants of blocks of size two are non-zero (i.e. if the distribution is a centered semicircular), then the distribution is invariant under $\mathring{O}_n^\varepsilon$.

Proof. Let x_1, \dots, x_n be ε -independent and identically distributed, let $k \in \mathbb{N}$ and let $j(1), \dots, j(k) \in \{1, \dots, n\}$.

Let $(\mathcal{C}, G) \in \{(P, S_n^\varepsilon), (P_{\text{even}}, H_n^\varepsilon), (P_{1,2}, \mathring{B}_n^\varepsilon), (P_2, \mathring{O}_n^\varepsilon)\}$ and let the cumulants of the distributions of x_1, \dots, x_n be according to the assumptions in (a), (b), (c) or (d) respectively. By the moment-cumulant formula (Proposition 3.5) and our assumptions on the cumulants we have:

$$\varphi(x_{j(1)} \cdots x_{j(k)}) = \sum_{\pi \in NC_{\mathcal{C}}^\varepsilon[j]} \kappa_\pi(x_{j(1)}, \dots, x_{j(k)})$$

For $\pi \in NC_{\mathcal{C}}^\varepsilon[j]$, the cumulant $\kappa_\pi(x_{j(1)}, \dots, x_{j(k)})$ factorizes according to the blocks of π (see [SW16]) and on each such block the indices $j(l)$ coincide, since $\pi \leq \ker j$. Now, x_1, \dots, x_n are identically distributed, thus those single block cumulants do not depend on the index j , and hence nor does κ_π . Therefore, we put $\kappa_\pi := \kappa_\pi(x_{j(1)}, \dots, x_{j(k)})$ for any j with $\pi \leq \ker j$. We then compute, using Proposition 8.8:

$$\begin{aligned} & \sum_{i(1), \dots, i(k)} \varphi(x_{i(1)} \cdots x_{i(k)}) u_{i(1)j(1)} \cdots u_{i(k)j(k)} \\ &= \sum_{i(1), \dots, i(k)} \left(\sum_{\pi \in NC_{\mathcal{C}}^\varepsilon[i]} \kappa_\pi(x_{i(1)}, \dots, x_{i(k)}) \right) u_{i(1)j(1)} \cdots u_{i(k)j(k)} \\ &= \sum_{i(1), \dots, i(k)} \left(\sum_{\pi \in P(k)} \delta_{\pi \in NC_{\mathcal{C}}^\varepsilon[i]} \kappa_\pi \right) u_{i(1)j(1)} \cdots u_{i(k)j(k)} \\ &= \sum_{\pi \in P(k)} \kappa_\pi \sum_{i(1), \dots, i(k)} \delta_{\pi \in NC_{\mathcal{C}}^\varepsilon[i]} u_{i(1)j(1)} \cdots u_{i(k)j(k)} \\ &= \sum_{\pi \in P(k)} \kappa_\pi \delta_{\pi \in NC_{\mathcal{C}}^\varepsilon[j]} \\ &= \varphi(x_{j(1)} \cdots x_{j(k)}) \end{aligned}$$

Thus, the distribution of x_1, \dots, x_n is invariant under G . \square

Remark 9.7. Distributional invariance under S_n^+ is also called *quantum exchangeability*, while invariance under S_n is called *exchangeability*. The former one implies the latter one [KS09]. In our case, invariance under S_n^ε (which we could call *ε -quantum exchangeability*) does *not* imply exchangeability. We only obtain invariance under T_n^ε (which we could call *ε -exchangeability*), i.e.:

$$\varphi(x_{j(1)} \cdots x_{j(k)}) = \varphi(x_{\sigma(j(1))} \cdots x_{\sigma(j(k))}) \quad \forall \sigma \in T_n^\varepsilon$$

Since exchangeability of variables implies that they are identically distributed, and since we only have ε -exchangeability in our case, it is likely that one can weaken the assumptions of our de Finetti theorem.

10. PARTITION CALCULUS AND INTERTWINERS

As already mentioned in Section 8, Woronowicz's concept of intertwiner spaces gives a Tannaka-Krein type approach to compact matrix quantum groups. Moreover, the calculus with intertwiners provides a fairly easy way of deducing C^* -algebraic relations from others. The concept of easy quantum groups as introduced by Banica and Speicher [BS09] offers yet another simplification of the intertwiner calculus: For any easy quantum group, its intertwiner space is spanned by maps indexed by partitions as in Definition 8.1. For ε -versions of easy quantum groups, the situation is a bit more delicate and we cannot give a partition approach in general. However, we may at least provide some access to the intertwiner spaces using modified partitions which we will now develop in three steps.

10.1. Expressing relations as intertwiners. Let us introduce certain linear maps extending Definition 8.2.

Definition 10.1. For $n \in \mathbb{N}$ we define the following linear maps from $(\mathbb{C}^n)^{\otimes 2}$ to $(\mathbb{C}^n)^{\otimes 2}$.

$$\begin{aligned} R^1_{\times}(e_i \otimes e_j) &:= \delta_{\varepsilon_{ij}=1} e_j \otimes e_i \\ R^1_{||}(e_i \otimes e_j) &:= \delta_{\varepsilon_{ij}=1} e_i \otimes e_j \\ R^0_{||}(e_i \otimes e_j) &:= \delta_{\varepsilon_{ij}=0} e_i \otimes e_j \\ R^0_{\sqcup}(e_i \otimes e_j) &:= \delta_{ij} \sum_k \delta_{\varepsilon_{ik}=0} e_k \otimes e_k \end{aligned}$$

We will now describe the intertwiners implementing relations such as R^ε and \mathring{R}^ε which we recall here (Def. 5.1, 6.1 and 6.6):

$$\begin{aligned}
(R^\varepsilon 1) \quad & u_{ik}u_{jl} = u_{jl}u_{ik} \text{ if } \varepsilon_{ij} = 1 \text{ and } \varepsilon_{kl} = 1 \\
(R^\varepsilon 2) \quad & u_{ik}u_{jl} = u_{jk}u_{il} \text{ if } \varepsilon_{ij} = 1 \text{ and } \varepsilon_{kl} = 0 \\
& \text{and } u_{ik}u_{jl} = u_{il}u_{jk} \text{ if } \varepsilon_{ij} = 0 \text{ and } \varepsilon_{kl} = 1 \\
(\mathring{R}^\varepsilon 2) \quad & \delta_{\varepsilon_{kl}=0}u_{ik}u_{jl} = \delta_{\varepsilon_{ij}=0}u_{ik}u_{jl} \\
(R^\varepsilon) \quad & (R^\varepsilon 1) \text{ and } (R^\varepsilon 2) \\
(\mathring{R}^\varepsilon) \quad & (R^\varepsilon 1) \text{ and } (\mathring{R}^\varepsilon 2) \\
(R_{\text{aut}}) \quad & \sum_k \delta_{\varepsilon_{kl}=1}u_{ik} = \sum_j \delta_{\varepsilon_{ij}=1}u_{jl}
\end{aligned}$$

Moreover, we define:

$$\begin{aligned}
(R'^\varepsilon 2) \quad & u_{ik}u_{jl} = \delta_{kl} \sum_m \delta_{\varepsilon_{km}=0}u_{jm}u_{im} \text{ if } \varepsilon_{ij} = 1 \text{ and } \varepsilon_{kl} = 0 \\
& \text{and } u_{ik}u_{jl} = \delta_{ij} \sum_m \delta_{\varepsilon_{im}=0}u_{ml}u_{mk} \text{ if } \varepsilon_{ij} = 0 \text{ and } \varepsilon_{kl} = 1 \\
(R'^\varepsilon) \quad & (R^\varepsilon 1) \text{ and } (R'^\varepsilon 2)
\end{aligned}$$

Lemma 10.2. *Let $G \subset O_n^+$ be a compact matrix quantum group.*

- (a) $R^1_{\chi} + R^0_{\begin{smallmatrix} | \\ | \end{smallmatrix}}$ is an intertwiner of G if and only if (R^ε) holds.
- (b) $R^1_{\chi} + R^0_{\begin{smallmatrix} \sqcup \\ \sqcap \end{smallmatrix}}$ is an intertwiner of G if and only if (R'^ε) holds.
- (c) R^1_{χ} is an intertwiner of G if and only if $(\mathring{R}^\varepsilon)$ holds.
- (d) $R^0_{\begin{smallmatrix} | \\ | \end{smallmatrix}}$ is an intertwiner of G if and only if $(\mathring{R}^\varepsilon 2)$ holds.
- (e) ε is an intertwiner of G (i.e. $u\varepsilon = \varepsilon u$) if and only if (R_{aut}) holds.

Proof. (a) We first compute:

$$\begin{aligned}
& u^{\otimes 2} \left(R^1_{\chi} + R^0_{\begin{smallmatrix} | \\ | \end{smallmatrix}} \right) (e_l \otimes e_k) \\
&= u^{\otimes 2} (\delta_{\varepsilon_{kl}=1} e_k \otimes e_l + \delta_{\varepsilon_{kl}=0} e_l \otimes e_k) \\
&= \sum_{i,j} (\delta_{\varepsilon_{kl}=1} u_{ik}u_{jl} + \delta_{\varepsilon_{kl}=0} u_{il}u_{jk}) \otimes e_i \otimes e_j
\end{aligned}$$

And:

$$\begin{aligned}
 & \left(R^1_{\chi} + R^0_{||} \right) u^{\otimes 2}(e_l \otimes e_k) \\
 &= \sum_{i,j} \left(u_{jl}u_{ik} \otimes R^1_{\chi}(e_j \otimes e_i) + u_{il}u_{jk} \otimes R^0_{||}(e_i \otimes e_j) \right) \\
 &= \sum_{i,j} \left(\delta_{\varepsilon_{ij}=1} u_{jl}u_{ik} + \delta_{\varepsilon_{ij}=0} u_{il}u_{jk} \right) \otimes e_i \otimes e_j
 \end{aligned}$$

We infer that $R^1_{\chi} + R^0_{||}$ is an intertwiner if and only if for all i, j, k, l :

$$\delta_{\varepsilon_{kl}=1} u_{ik}u_{jl} + \delta_{\varepsilon_{kl}=0} u_{il}u_{jk} = \delta_{\varepsilon_{ij}=1} u_{jl}u_{ik} + \delta_{\varepsilon_{ij}=0} u_{il}u_{jk}$$

These relations are equivalent to R^ε .

(b) We proceed like in (a) by computing:

$$\begin{aligned}
 & u^{\otimes 2} \left(R^1_{\chi} + R^0_{\sqcup} \right) (e_l \otimes e_k) \\
 &= u^{\otimes 2} \left(\delta_{\varepsilon_{kl}=1} e_k \otimes e_l + \delta_{kl} \sum_m \delta_{\varepsilon_{km}=0} e_m \otimes e_m \right) \\
 &= \sum_{i,j} \left(\delta_{\varepsilon_{kl}=1} u_{ik}u_{jl} + \delta_{kl} \sum_m \delta_{\varepsilon_{km}=0} u_{im}u_{jm} \right) \otimes e_i \otimes e_j
 \end{aligned}$$

And:

$$\begin{aligned}
 & \left(R^1_{\chi} + R^0_{\sqcup} \right) u^{\otimes 2}(e_l \otimes e_k) \\
 &= \sum_{i,j} u_{jl}u_{ik} \otimes R^1_{\chi}(e_j \otimes e_i) + \sum_{p,m} u_{pl}u_{mk} \otimes R^0_{\sqcup}(e_p \otimes e_m) \\
 &= \sum_{i,j} \delta_{\varepsilon_{ij}=1} u_{jl}u_{ik} \otimes e_i \otimes e_j + \sum_{p,m,i} \delta_{pm} \delta_{\varepsilon_{im}=0} u_{pl}u_{mk} \otimes e_i \otimes e_i \\
 &= \sum_{i,j} \delta_{\varepsilon_{ij}=1} u_{jl}u_{ik} \otimes e_i \otimes e_j + \sum_{i,m} \delta_{\varepsilon_{im}=0} u_{ml}u_{mk} \otimes e_i \otimes e_i \\
 &= \sum_{i,j} \left(\delta_{\varepsilon_{ij}=1} u_{jl}u_{ik} + \delta_{ij} \sum_m \delta_{\varepsilon_{im}=0} u_{ml}u_{mk} \right) \otimes e_i \otimes e_j
 \end{aligned}$$

Assertion (c) is the contents of Lemma 8.3, and (d) and (e) are straightforward. \square

We infer that any of the above relations in Lemma 10.2 may be implemented by intertwiners. Hence, each of them passes through the comultiplication map Δ of compact matrix quantum groups. This means, that we may define quantum groups satisfying these relations. In this sense, the above Lemma 10.2 gives a more systematic proof of Lemma 5.2 and Lemma 6.4. Moreover, using the building blocks $(R^\varepsilon 1)$, $(\mathring{R}^\varepsilon 2)$, $(R^\varepsilon 2)$ and (R_{aut}) , we may define possibly new quantum subgroups of O_n^+ by quotienting out the following relations:

- only $(\mathring{R}^\varepsilon 2)$
- only (R_{aut})
- $(\mathring{R}^\varepsilon 2)$ together with (R_{aut})
- $(\mathring{R}^\varepsilon)$ together with (R_{aut})
- (R^ε) together with (R_{aut})

It is not clear, whether the relations $(R^\varepsilon 1)$ and $(R^\varepsilon 2)$ may be expressed separately by intertwiners, so we don't know whether we may define quantum groups by using these relations separately.

10.2. Equivalence of relations by intertwiner calculus. Having expressed our relations by intertwiners, it is very easy to deduce some relations from others or even to show their equivalence. All we need to show is that we may construct certain intertwiners from others using the operations of intertwiner spaces [Wor88]:

- If S and T are intertwiners of G , so are $S \otimes T$, ST and T^* , as well as $\alpha S + \beta T$, for $\alpha, \beta \in \mathbb{C}$.
- The identity map $\text{id} : \mathbb{C}^n \rightarrow \mathbb{C}^n$ is an intertwiner of G .
- If $G \subset O_n^+$, then the maps T_{\sqcap} and T_{\sqcup} are intertwiners of G (see for instance [Web13, Lemma 2.5]).

Recall from Definition 8.1 the definition of T_π , where π is some partition.

Lemma 10.3. *Let $G \subset O_n^+$ be a quantum subgroup of O_n^+ . Concerning intertwiners in G , we have:*

- (a) R_{χ}^1 is an intertwiner if and only if $R_{||}^0$ and $R_{\chi}^1 + R_{||}^0$ are.
- (b) $R_{\chi}^1 + R_{||}^0$ is an intertwiner if and only if $R_{\chi}^1 + R_{\sqcup}^0$ is.
- (c) $R_{||}^0$ is an intertwiner if and only if R_{\sqcup}^0 is.
- (d) Let $T_{\mathbb{H}}$ be an intertwiner of G .
Then $R_{\chi}^1 + R_{||}^0$ is an intertwiner if and only if R_{χ}^1 is.
- (e) Let T_{\top} be an intertwiner of G .

Then $R_{||}^0$ is an intertwiner if and only if ε is.

Proof. (a) We have:

$$R_{||}^0 = (\text{id} \otimes \text{id}) - \left(R_{\chi}^1\right)^2$$

(b) We compute:

$$\begin{aligned} & (T_{\sqcup} \otimes \text{id} \otimes \text{id}) \left(\text{id} \otimes \left(R_{\chi}^1 + R_{||}^0 \right) \otimes \text{id} \right) (\text{id} \otimes \text{id} \otimes T_{\sqcap}) (e_i \otimes e_j) \\ &= \sum_k (T_{\sqcup} \otimes \text{id} \otimes \text{id}) \left(\text{id} \otimes \left(R_{\chi}^1 + R_{||}^0 \right) \otimes \text{id} \right) (e_i \otimes e_j \otimes e_k \otimes e_k) \\ &= \sum_k (T_{\sqcup} \otimes \text{id} \otimes \text{id}) (\delta_{\varepsilon_{kj}=1} e_i \otimes e_k \otimes e_j \otimes e_k + \delta_{\varepsilon_{kj}=0} e_i \otimes e_j \otimes e_k \otimes e_k) \\ &= \sum_k \delta_{\varepsilon_{kj}=1} \delta_{ik} e_j \otimes e_k + \sum_k \delta_{\varepsilon_{kj}=0} \delta_{ij} e_k \otimes e_k \\ &= \left(R_{\chi}^1 + R_{\sqcup}^0 \right) (e_i \otimes e_j) \end{aligned}$$

Thus, if $R_{\chi}^1 + R_{||}^0$ is an intertwiner, so is $R_{\chi}^1 + R_{\sqcup}^0$. A similar computation shows the converse.

(c) Omitting R_{χ}^1 in the proof of (b), we obtain the proof of (c).

(d) If $R_{\chi}^1 + R_{||}^0$ is an intertwiner of G , then $R_{\chi}^1 + R_{\sqcup}^0$, too, by (b). From

$$\left(R_{\chi}^1 + R_{\sqcup}^0 \right) T_{\sqcap} = R_{\sqcap}^0$$

and (c), we infer that $R_{||}^0$ is an intertwiner of G .

(e) If $R_{||}^0$ is an intertwiner of G , then R_{\sqcup}^0 , too, by (c). We compute:

$$T_{\sqcap} R_{\sqcup}^0 T_{\sqcap}^* (e_i) = \sum_k \delta_{\varepsilon_{ik}=0} e_k$$

Thus, $\varepsilon = \text{id} - T_{\sqcap} R_{\sqcup}^0 T_{\sqcap}^*$ is an intertwiner of G . Conversely, if ε is an intertwiner of G , then also

$$R_{\sqcup}^0 = T_{\sqcap}^* (\text{id} - \varepsilon) T_{\sqcap}$$

is an intertwiner of G . □

Using the above lemma, we immediately see the consequences for the relations, recovering results from Lemma 6.2 and Lemma 6.7.

Lemma 10.4. *Let $G \subset O_n^+$ be a quantum subgroup of O_n^+ .*

- (a) *The relations (\dot{R}^ε) imply (R^ε) .*
- (b) *The relations (R^ε) hold if and only if the relations (R'^ε) hold.*
- (c) *If $u_{ik}u_{jk} = u_{ki}u_{kj} = 0$ for all $i \neq j$, then (\dot{R}^ε) and (R^ε) are equivalent.*
- (d) *If $G \subset S_n^+$, then $(\dot{R}^{\varepsilon 2})$ and (R_{aut}) are equivalent.*

Proof. The relations $u_{ik}u_{jk} = u_{ki}u_{kj} = 0$ for all $i \neq j$ are equivalent to the fact that T_{H} is an intertwiner (see for instance [Web13, Lemma 2.5]). Moreover, one can check that T_{H} is an intertwiner if and only if $G \subset S_n^+$. We then use Lemma 10.3 together with Lemma 10.2. \square

10.3. Further relations that hold in O_n^ε .

Proposition 10.5. *The relations R'^ε hold in O_n^ε . In particular:*

$$\varepsilon_{ij} = 1, \varepsilon_{kl} = 0, k \neq l: \quad u_{ik}u_{jl} = 0 \text{ and } u_{ki}u_{lj} = 0$$

$$\begin{aligned} \varepsilon_{ij} = 1: \quad & \sum_{m \neq k \text{ and } \varepsilon_{km}=0} u_{im}u_{jm} = \sum_{m=k \text{ or } \varepsilon_{km}=1} u_{im}u_{jm} = 0 \\ & \sum_{m \neq k \text{ and } \varepsilon_{km}=0} u_{mi}u_{mj} = \sum_{m=k \text{ or } \varepsilon_{km}=1} u_{mi}u_{mj} = 0 \end{aligned}$$

Proof. We use Lemma 10.3(b) and deduce from R'^ε and R^ε for $\varepsilon_{ij} = 1$:

$$u_{ik}u_{jk} = \sum_{m \neq k: \varepsilon_{km}=0} u_{im}u_{jm} + u_{ik}u_{jk}$$

Together with $\sum_m u_{im}u_{jm} = \delta_{ij}$, this proves the claim. \square

The above relations for sums of $u_{im}u_{jm}$ seem a bit strange at first glance. However, note that such groupings of summands are nothing unusual in the theory of quantum groups. Indeed, while we have that the sum $\sum_m u_{im}u_{jm}$ is zero for $i \neq j$ in O_n^+ , we require that all of its summands $u_{im}u_{jm}$ are zero in S_n^+ and H_n^+ . Now, the requirement in O_n^ε is something in between: certain subsums have to be zero.

10.4. First ideas for a partition calculus for O_n^ε . In Section 10.2, we saw the use of intertwiner calculus for compact matrix quantum groups. For easy quantum groups [BS09], this intertwiner calculus can be transferred to a partition calculus – the intertwiners of an easy quantum group G are spanned by linear maps T_π indexed by partitions π , and we have: If T_π and T_σ are intertwiners of G , so are $T_{\pi \otimes \sigma}$, $T_{\pi \sigma}$

and T_{π^*} . See [BS09] or [Web13] for details. We will now develop a pictorial approach to intertwiners of O_n^ε with the help of which some of the intertwiner calculus of the preceding subsection can be done by purely pictorial means.

The relations R^ε are implemented by the intertwiner $R^1 \chi + R^0 \parallel$ which is somehow a superposition of the maps T_χ and T_{\parallel} – depending on the ε -entry, we either apply T_χ or T_{\parallel} . We therefore propose the following symbolism:

$$\begin{aligned} S \boxed{\chi} &:= R^1 \chi + R^0 \parallel \\ S \boxed{\chi} &:= R^1 \chi + R^0 \sqcup \\ S \boxed{\chi} &:= R^1 \parallel + R^0 \sqcup \end{aligned}$$

Note that $R^1 \parallel + R^0 \parallel = \text{id} \otimes \text{id}$. With this graphical calculus, the proof of Lemma 10.3(b) reads as follows, following the black lines inside the box for $\varepsilon_{ij} = 1$ and the gray ones for $\varepsilon_{ij} = 0$:

$$\boxed{\chi} = \begin{array}{c} \begin{array}{|c|c|} \hline \text{ } & \text{ } \\ \hline \end{array} \\ \begin{array}{|c|c|} \hline \text{ } & \text{ } \\ \hline \end{array} \end{array} = \begin{array}{c} \begin{array}{|c|c|} \hline \text{ } & \text{ } \\ \hline \end{array} \\ \begin{array}{|c|c|} \hline \text{ } & \text{ } \\ \hline \end{array} \end{array}$$

Moreover, we easily see on the graphical level (the composition of linear maps meaning that we put one picture above the other and follow the lines):

$$\begin{aligned} S \boxed{\chi} S \boxed{\chi} &= T_{\parallel} \\ S \boxed{\chi} S \boxed{\chi} &= S \boxed{\chi} S \boxed{\chi} = S \boxed{\chi} \end{aligned}$$

But note that:

$$S \begin{array}{|c|} \hline \text{X} \\ \hline \end{array} S \begin{array}{|c|} \hline \text{X} \\ \hline \end{array} (e_i \otimes e_j) = R^1_{||} (e_i \otimes e_j) + \delta_{ij} \sum_k \left(\sum_l \delta_{\varepsilon_{il}=0} \delta_{\varepsilon_{lk}=0} \right) e_k \otimes e_k$$

Thus:

$$S \begin{array}{|c|} \hline \text{X} \\ \hline \end{array} S \begin{array}{|c|} \hline \text{X} \\ \hline \end{array} \neq S \begin{array}{|c|} \hline \square \\ \hline \end{array}$$

This can also be seen on the graphical level: The composition of the gray lines yields a loop. Therefore, we need to extend our pictures by keeping track of the number of gray loops, and our pictorial representation explodes – just like the intertwiner space itself. This seems to put an end to the attempt of describing the intertwiner spaces of O_n^ε graphically. However, depending on ε , the number of loops needed in an extended definition of $S \begin{array}{|c|} \hline \text{X} \\ \hline \end{array}$ might be limited. Indeed, if ε is as in

Example 3.2(f), then:

$$\sum_l \delta_{\varepsilon_{il}=0} \delta_{\varepsilon_{lk}=0} = \delta_{\varepsilon_{ik}=0} + \delta_{ik} + 1$$

Therefore $S \begin{array}{|c|} \hline \text{X} \\ \hline \end{array} S \begin{array}{|c|} \hline \text{X} \\ \hline \end{array}$ is again in the span of the already known intertwiners, since:

$$S \begin{array}{|c|} \hline \text{X} \\ \hline \end{array} S \begin{array}{|c|} \hline \text{X} \\ \hline \end{array} = S \begin{array}{|c|} \hline \square \\ \hline \end{array} + T \begin{array}{|c|} \hline \text{H} \\ \hline \end{array} + T \begin{array}{|c|} \hline \sqcup \\ \hline \end{array}$$

11. OPEN QUESTIONS

We want to finish this articles with a list of open questions for future work on ε -mixed quantum groups.

11.1. About $\mathring{O}_n^\varepsilon$ and the choice $\varepsilon_{ii} = 0$. Recall that $\mathring{S}_n^\varepsilon = S_n^\varepsilon$ and $\mathring{H}_n^\varepsilon = H_n^\varepsilon$, while $\mathring{O}_n^\varepsilon \neq O_n^\varepsilon$ and $\mathring{B}_n^\varepsilon \neq B_n^\varepsilon$ in general. We therefore ask: For which matrices ε do we have $\mathring{O}_n^\varepsilon \neq O_n^\varepsilon$? Recall from Section 8 that $\mathring{O}_n^\varepsilon = O_n^\varepsilon = O_n^+$ for $\varepsilon = \varepsilon_{\text{free}}$, but $\mathring{O}_n^\varepsilon = H_n \neq O_n = O_n^\varepsilon$ for $\varepsilon = \varepsilon_{\text{comm}}$. This is due to some subtlety with respect to the choice of the diagonal entries of ε : The decision to put $\varepsilon_{ii} = 0$ or $\varepsilon_{ii} = 1$ does not affect the definition of ε -independence, nor the relations R^ε (note that the intertwiner $R^1_{\text{X}} + R^0_{||}$ does not depend on the choice of the diagonal entries). However, the choice of the cumulants in [SW16] is affected by the choice of the diagonal entries, and putting them to zero is the choice for the free cumulants rather than for the classical ones. Thus, in some cases, our choice of the diagonal entries fits better with the free situation – as may be seen with the relations \mathring{R}^ε and the discussion on

$\mathring{O}_n^\varepsilon$. Note that the intertwiner R_χ^1 , which implements the relations \mathring{R}^ε , *does* depend on the choice of the diagonal entries.

We conclude that depending on the choice of the diagonal entries, we have yet another possible definition of the relations \mathring{R}^ε , and yet another possible definition of $\mathring{O}_n^\varepsilon$.

Note that in case $\mathring{O}_n^\varepsilon \neq O_n^\varepsilon$, the quantum group $\mathring{O}_n^\varepsilon$ acts on the ε -sphere $S_{\mathbb{R},\varepsilon}^{n-1}$, but the action is not maximal. On which quantum space does it act maximally? We don't know. It would be interesting to analyze the spheres with the relations $x_i x_j = 0$ for $\varepsilon_{ij} = 0$, either combined with $x_i x_j = x_j x_i$ for $\varepsilon_{ij} = 1$ or separately.

Furthermore, Lemma 6.4 allows us to define yet another variant of O_n^ε by taking the quotient of O_n^+ by the relations $(\mathring{R}^\varepsilon 2)$ of Definition 6.1. This time, we obtain a quantum group which contains O_n^ε as a subgroup, and we might want to study this quantum group, too. See also the end of Section 10.1 for further possible definitions of quantum groups.

11.2. About S_n^ε . We know from Proposition 5.6, that O_n^ε is always noncommutative. However, we have seen in Examples 6.10 and 6.11, that this is not always the case for S_n^ε . Can we characterize the noncommutativity of S_n^ε in terms of properties of ε ? This question is deeply linked with the investigations of graphs having no quantum symmetry, as treated in [BB07] and [Ful06]. Concerning the link to Banica's quantum automorphism groups $S_n^{\Gamma_\varepsilon}$ of graphs Γ_ε , it would be interesting to find examples such that $T_n^\varepsilon \neq S_n^\varepsilon \neq S_n^{\Gamma_\varepsilon}$. Moreover, do we have $S_n^\varepsilon \neq S_n^{\varepsilon'}$ for $\varepsilon \neq \varepsilon'$ in analogy to the case of O_n^ε (see Proposition 5.6)?

Also, it would be interesting to know whether or not $C(S_5^\varepsilon)$ is commutative for ε as in Example 3.2(f), since this matrix cannot be obtained from iterated grouping of independences. Likewise we would like to know whether S_6^ε is nontrivial for ε as in Example 7.3(c). Here, T_6^ε is the trivial group, so the question is, whether we may find some nontrivial quantum group S_n^ε such that T_n^ε is trivial. Finally, in [BB09], Banica and Bichon classify all quantum subgroups of S_4^+ . It would be interesting to know which of them are of the type S_n^ε as studied in our article, and which are not.

11.3. About T_n^ε . Can we read off the group $T_n^\varepsilon \subset S_n$ of Definition 7.1 from the matrix ε ? Can we link it to the Coxeter group \mathbb{Z}_2^ε ? How does T_n^ε look like for iterated grouping of variables as in Example 3.2? What is T_n^ε for ε as in Example 3.2(f)?

Furthermore, it would be interesting to see whether T_n^ε is a kind of an invariant for O_n^ε or S_n^ε . Note that the statement $O_n^\varepsilon \neq O_n^{\varepsilon'}$ (or

$S_n^\varepsilon \neq S_n^{\varepsilon'}$) is a rather weak one since this only means that there is no $*$ -isomorphism between the associated C^* -algebras sending generators to generators. What about other isomorphisms? Can they be excluded with the help of T_n^ε ?

11.4. Extensions of the de Finetti theorem. Recapturing the proof of Theorem 9.6, we observe that the crucial point is Proposition 8.8: Is $\delta_{\pi \in NC_{\mathcal{C}}^\varepsilon[i]}$ an intertwiner of G ? The input data for proving it is:

- (i) T_π is an intertwiner of G , for $\pi \in \mathcal{C}$ a noncrossing partition.
- (ii) R_χ^1 is an intertwiner of G .

Now, for quantum groups \dot{G}_n^ε , item (ii) is true, by Lemma 8.3. Thus, from a de Finetti point of view, \dot{O}_n^ε is more natural than O_n^ε . On the other hand, O_n^ε is more natural from a quantum action point of view (Theorem 5.7). Hence, it would be interesting to find de Finetti theorems also for R^ε , in particular for O_n^ε and B_n^ε .

11.5. A partition calculus for ε -quantum groups. As sketched in Section 10.4, a general partition calculus for O_n^ε seems hopeless, since the graphical calculus seems to explode. This fits (in the easy quantum group philosophy) with the fact that the quantum group containing the maximal intertwiner space, namely T_n^ε , can be trivial (which means that its intertwiner spaces consists of all possible linear maps). however, it would be interesting to study the intertwiner spaces in special examples of ε . Note that for $\varepsilon = \varepsilon_{\text{comm}}$ and $\varepsilon = \varepsilon_{\text{free}}$ we do have such a partition calculus, since $S \begin{array}{|c|} \hline \times \\ \hline \end{array} = T \chi$ in the former case and $S \begin{array}{|c|} \hline \times \\ \hline \end{array} = T \parallel$ in the latter. Moreover, Example 3.2(f) seems to be promising for providing an example with an accessible intertwiner space structure.

However, we need to define pictures also for more general intertwiners. The idea is to take any partition $\pi \in P(k, l)$ and to replace each crossing by one of the basic boxes such as:

$$\begin{array}{|c|} \hline \times \\ \hline \end{array}, \begin{array}{|c|} \hline \times \\ \hline \end{array}, \begin{array}{|c|} \hline \times \\ \hline \end{array}, \text{ loops } \dots$$

In order to have a partition approach to the intertwiners of quantum subgroups of O_n^ε , for instance for S_n^ε , the situation is even more involved: We need to come up with symbols for partitions with blocks of arbitrary sizes – and it is not even clear how to define crossings for such partitions.

In any case, any progress with respect to the decription of the intertwiner spaces of O_n^ε or S_n^ε will certainly extend our understanding of quantum subgroups of O_n^+ .

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